Decomposition of Besov Spaces

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1. Introduction. In the last decade, many function or distribution spaces have been found to admit a decomposition, in the sense that every member of the space is a linear combination of basic functions of a particularly elementary form. Such decompositions simplify the analysis of the spaces and the operators acting on them. Here we obtain two types of decompositions for distributions in the homogeneous Besov spaces $\dot{B}_p^{\alpha q}$, $-\infty < \alpha < +\infty$, $0 < p, q \leqq +\infty$, and present some applications of these results.

Defining the Fourier transform by $\hat{f}(\xi) = \int f(x) e^{-ix\cdot\xi} dx$, let $\{\varphi_\nu\}_{\nu\in\mathbf{Z}}$ be a family of functions on $\mathbf{R}^n$ satisfying

$$\varphi_\nu \in \mathcal{S}, \tag{1.1}$$

$$\operatorname{supp} \hat{\varphi}_\nu \subseteq \left\{\xi \in \mathbf{R}^n : \frac{1}{2} \leqq 2^{-\nu}|\xi| \leqq 2\right\}, \tag{1.2}$$

$$|\hat{\varphi}_\nu(\xi)| \geqq c > 0 \text{ if } \frac{3}{5} \leqq 2^{-\nu}|\xi| \leqq \frac{5}{3}, \tag{1.3}$$

and

$$|\partial^\gamma \hat{\varphi}_\nu(\xi)| \leqq c_\gamma 2^{-\nu|\gamma|} \qquad \text{for every multi-index } \gamma. \tag{1.4}$$

The Besov space $\dot{B}_p^{\alpha q}$, $-\infty < \alpha < +\infty$, $0 < p, q \leqq +\infty$, is the collection of all $f \in \mathcal{S}'/\mathcal{P}$ (tempered distributions modulo polynomials) such that

$$\|f\|_{\dot{B}_p^{\alpha q}} = \left(\sum_{\nu\in\mathbf{Z}} (2^{\nu\alpha}\|\varphi_\nu * f\|_{L^p})^q\right)^{1/q} < +\infty,$$

with the usual interpretation if $q = +\infty$. This definition is independent of the family $\{\varphi_\nu\}$ satisfying (1.1–4); see [20], p. 64.

Our purpose is to show that each $f \in \dot{B}_p^{\alpha q}$ can be decomposed into a sum of simple building blocks. The building blocks in our first decomposition are similar to the atoms in the atomic decomposition of Hardy spaces $H^p(\mathbf{R}^n)$, $0 < p \leqq 1$ ([10], [18], [6], [30]). We define an (α,p)-atom $a(x)$ ($-\infty < \alpha < +\infty$, $0 < p \leqq +\infty$) to be a function satisfying, for some cube $Q \subseteq \mathbf{R}^n$,

$$\operatorname{supp} a \subseteq 3Q, \tag{1.5}$$

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