

Characterization of Complex Manifolds by the Isotropy Subgroups of Their Automorphism Groups

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§0. An *automorphism* $\alpha: M \rightarrow M$ of a complex manifold M is by definition a holomorphic bijective map of M to itself. The inverse of such a map is also holomorphic (cf. [18] for the only nontrivial point, that α is necessarily everywhere nonsingular); and the set of all such mappings forms a group, denoted by $\text{Aut}(M)$. This group need not be a Lie group. For instance, $\text{Aut}(\mathbb{C}^2)$ contains the maps $(z, w) \rightarrow (z, w + f(z))$, where $f: \mathbb{C} \rightarrow \mathbb{C}$ is an arbitrary holomorphic function; and it is easy to see that this set of maps is not contained in a (finite-dimensional) Lie group. For one thing, this set of maps is infinite-dimensional in any reasonable sense of that phrase.

If M is a Riemann surface, then $\text{Aut}(M)$ is in fact a Lie group. This can be verified directly for \mathbb{CP}^1 , \mathbb{C} , and the unit disc in \mathbb{C} , and follows for other Riemann surfaces by uniformization considerations: $\text{Aut}(M)$ is a closed subgroup of the automorphism group of the simply-connected cover of M . The nature of $\text{Aut}(M)$, M a Riemann surface, was the subject of considerable investigation classically. The subject of the present paper is a generalization of one such classical result, namely a result concerning the subgroup of automorphisms which fix a given point $p \in M$.

If M is again a complex manifold, then for each $p \in M$, the set $\{\alpha \in \text{Aut}(M) : \alpha(p) = p\}$ is a subgroup of M , called the *isotropy subgroup* of p and denoted by I_p . Again, I_p need not be a Lie group; but, in case $\text{Aut}(M)$ is a Lie group, I_p is a closed subgroup and is hence also a Lie group.

If M is a bounded, connected open subset of \mathbb{C} , then I_p is a compact Lie group. This again follows from uniformization: Let $\Delta \xrightarrow{\pi} M$ be the universal cover of M , $\Delta =$ the unit disc. Choose $\tilde{p} \in \pi^{-1}(p)$. Then each $\alpha \in I_p$ lifts to a unique element $\tilde{\alpha} \in I_{\tilde{p}} \subset \text{Aut}(\Delta)$. The group $I_{\tilde{p}}$ is a compact Lie group isomorphic to the unit circle (under multiplication) via the map $\beta \rightarrow d\beta|_{\tilde{p}}$, $\beta \in I_{\tilde{p}}$. It follows that I_p is a compact Lie group and that the map $\alpha \rightarrow d\alpha|_p$ is a group isomorphism of I_p onto a closed subgroup of the unit circle multiplicative group. It seems to have been first discovered by Bruné [2] that this subgroup is finite except when M is biholomorphic to Δ . (Of course, a closed subgroup of the unit circle is either finite or the whole