

The Index of an Elementary Operator

LAWRENCE A. FIALKOW

1. Introduction. Let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on a separable infinite dimensional complex Hilbert space \mathcal{H} . Let $A = (A_1, \dots, A_n)$ and $B = (B_1, \dots, B_n)$ denote commutative n -tuples of operators in $\mathcal{L}(\mathcal{H})$ and let $R \equiv R_{AB} : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{H})$ denote the *elementary operator* defined by $R(X) = A_1XB_1 + \dots + A_nXB_n$. If $(\mathcal{F}, \|\cdot\|)$ is a *norm ideal* in $\mathcal{L}(\mathcal{H})$ in the sense of [22], [34], then \mathcal{F} is invariant under R , and $R_{\mathcal{F}} \equiv R_{AB}|_{\mathcal{F}}$ is a bounded operator on \mathcal{F} . Commencing with G. Lumer and M. Rosenblum [28], several authors have studied spectral properties of R or $R_{\mathcal{F}}$ in terms of spectral invariants of A and B (or their operator components) [5], [8], [9], [13], [15], [16], [17], [18], [19], [20]. As a special case of interest in the sequel, let $A_0, B_0 \in \mathcal{L}(\mathcal{H})$, let f_1, \dots, f_n denote complex functions analytic in a neighborhood of $\sigma(A_0)$ (the spectrum of A_0), and let g_1, \dots, g_n denote complex functions analytic in a neighborhood of $\sigma(B_0)$. With $A_i = f_i(A_0)$ and $B_i = g_i(B_0)$ ($1 \leq i \leq n$) (as defined by the Riesz-Dunford functional calculus [31, Chapter 2]), we refer to the elementary operator R_{AB} as an *analytic elementary operator*; in [10, Theorem 10] G. Lumer and M. Rosenblum showed that the spectrum of R_{AB} is given by

$$(1.1) \quad \sigma(R_{AB}) = \left\{ \sum_{i=1}^n f_i(\alpha)g_i(\beta) : \alpha \in \sigma(A_0), \beta \in \sigma(B_0) \right\}.$$

For the general case, the spectrum of R was described by R. Harte [24] and an alternate description was given by R. Curto [8] (see (1.3)–(1.5) below). A. Carrillo and C. Hernandez [5], R. Curto [9], and the author [20] recently formulated various expressions for $\sigma_e(R_{AB})$, the Fredholm essential spectrum of R_{AB} (see (1.7)–(1.11) below). In the present note we study the problem of computing the index invariant $\text{ind}(R - \lambda)$ ($\lambda \in \mathbb{C} \setminus \sigma_e(R)$). The approach we take extends the methods used previously to compute the index invariants of the generalized derivations T_{AB} ($T_{AB}(X) = AX - XB$) [17, Theorem 4.2] and of the elementary multiplications S_{AB} ($S_{AB}(X) = AXB$) [18, Theorem 3.8]. In (1.13) we describe a condition on λ, A, B that is sufficient to obtain the expression for $\text{ind}(R - \lambda)$ that we present in Theorem 1.1. Section 2 contains a proof of this result. In Section 3 we show that if R is an analytic elementary operator, then (1.13) is satisfied for every $\lambda \in \mathbb{C} \setminus \sigma_e(R)$. In this case, instead of using Taylor spectra and the multivariate methods of Section 2, we develop the index formula using Harte spectra and a reduction to the single variable case ($n = 1$).