

The Comass Ball in $\Lambda^3(\mathbf{R}^6)^*$

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1. Introduction. A form ϕ on a Riemannian manifold which is d -closed and has comass one is called a calibration. Associated with a calibration is a geometry of submanifolds consisting of those submanifolds on which ϕ is a volume form. Such ϕ -submanifolds are homologically volume minimizing. Several new examples of calibrated geometries were introduced in Harvey-Lawson [3] (also see the survey [2]).

The important case of constant coefficient calibrations $\phi \in \Lambda^p(\mathbf{R}^n)^*$ on euclidean space is far from understood. The grassmannian $G(p, \mathbf{R}^n)$ of oriented (unit) p -planes in \mathbf{R}^n is naturally identified with the compact submanifold of $\Lambda^p \mathbf{R}^n$ consisting of all simple (decomposable) p -vectors ξ of unit length. A form ϕ has comass $\|\phi\|^*$ equal to the supremum of ϕ over $G(p, \mathbf{R}^n)$. Thus the comass ball $K^* \equiv \{\phi : \|\phi\|^* \leq 1\}$ is dual to the convex hull K of $G(p, \mathbf{R}^n)$. The set K is referred to as the *mass ball*. In terms of the associated hyperplane $\{\xi : \phi(\xi) = 1\}$, ϕ has comass one if and only if $G(p, \mathbf{R}^n)$ lies in the half-space $\{\xi : \phi(\xi) \leq 1\}$, and intersects the hyperplane $\{\xi : \phi(\xi) = 1\}$.

If ϕ has comass one then the contact set

$$\{\xi \in G(p, \mathbf{R}^n) : \phi(\xi) = 1\}$$

will be denoted $G(\phi)$ and sometimes referred to as the ϕ -grassmannian.

The purpose of this paper is to describe the comass ball K^* in the simplest non-classical case, $p = 3$, $n = 6$. Part of the description is available from Dadok-Harvey [1] and Morgan [5] and [6], and part of the description is new. Each form $\psi \in \Lambda^3(\mathbf{R}^6)^*$ is orthogonally equivalent to a form ϕ in the five-dimensional space V^5 consisting of all forms:

$$\phi(\lambda, \mu) \equiv \lambda_0 e_{123}^* + \lambda_1 e_{156}^* + \lambda_2 e_{426}^* + \lambda_3 e_{453}^* + \mu e_{456}^*.$$

Theorem 3.8 describes the intersection of the comass ball K^* with this section V^5 . There are two useful preliminary subcases. First, suppose one of the $\lambda_j = \pm 1$, say $\lambda_0 = 1$. Then a formula for the forms $\phi(1, \lambda_1, \lambda_2, \lambda_3, \mu)$ of comass at most one is given in Dadok-Harvey [1]. Second, Theorem 2.20 treats the subcase $\mu = 0$ and gives a formula for the comass of the form $\phi(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ to be at most 1. Namely, $\|\phi(\lambda)\|^* \leq 1$ if and only if $\max|\lambda_j| \leq 1$ and

$$2 - \lambda_0^2 - \lambda_1^2 - \lambda_2^2 - \lambda_3^2 - 2\lambda_0\lambda_1\lambda_2\lambda_3 + 2[(1 - \lambda_0^2)(1 - \lambda_1^2)(1 - \lambda_2^2)(1 - \lambda_3^2)]^{1/2} \geq 0.$$