

First Order Partial Differential Equations with Singular Solution

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Introduction. In classical treatises on first order PDE (e.g. Forsythe or Courant-Hilbert, v. 2) the discussions of first order PDE with singular solutions are informal. The objectives of this paper are, to intrinsically characterize this class of PDE, to prove a rigorous existence and uniqueness theorem for solutions (this is non-trivial in that the characteristic distribution vanishes), and to show how differential geometric invariants associated with the PDE determine properties of solutions.

The geometric theory of partial differential equations views a PDE as a locus of points in an appropriate jet bundle, (usually this locus is assumed to be the image of a smooth map $\mathcal{E}: E \rightarrow J$). Here we consider the simplest case of a single first order PDE for a real-valued function on a smooth manifold M , (i.e. $\mathcal{E}: E \rightarrow J^1(M, \mathbf{R})$ is a hypersurface). The interaction of this hypersurface with natural structures on $J^1(M, \mathbf{R})$ determines properties of the PDE and its solutions. Our first objective is to characterize PDE with singular solutions from this viewpoint. The presentation will be axiomatic. (The reader may prefer to take the normal form of Proposition 1.4 as a definition.) Next we will apply a theorem of symplectic geometry (due to Kostant-Sternberg) to yield existence and uniqueness of solutions near the singular solution. These solutions will be given as Lagrangian submanifolds of T^*M . Finally we will extract from the PDE a non-degenerate pseudo Riemannian metric $\langle \cdot, \cdot \rangle$ on M and show that the metric induced on smooth "initial data" $i: N \rightarrow (M, \langle \cdot, \cdot \rangle)$ determines two basic properties of the associated solution: first, the induced metric determines the Hessian at the critical locus of the solution and second, intrinsic differential geometric invariants of $(N, i^*\langle \cdot, \cdot \rangle)$ determine when the cotangent projection of the solution has a simple fold singularity (i.e. an $S_{1,0}$ singularity, see Appendix 1). The paper concludes with three examples. The first two are the classical examples of a PDE with singular solution. The last constructs PDE with singular solutions on any Lie group.

I. Graphlike PDE with singular solution. Given M a smooth manifold of dimension m , $C^\infty(M, \mathbf{R})$, $\mathfrak{X}(M)$ and $\Omega(M)$ will denote the smooth functions, vector fields and 1-forms on M . A subscript $x \in M$ will denote germs of such objects at x . $\mathcal{M}_x(M) \subset C_x^\infty(M, \mathbf{R})$ will denote the maximal ideal in the ring of germs