

Estimates for the Bochner-Riesz Operator with Negative Index

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Introduction. The purpose of this paper is to determine certain $(L^p(\mathbf{R}^n), L^q(\mathbf{R}^n))$ estimates for the Bochner-Riesz operator with negative index and $n \geq 2$. These operators are defined on $\mathcal{S}(\mathbf{R}^n)$, the class of rapidly decreasing C^∞ functions, by the equation

$$(T_\alpha f)^\wedge(\xi) = m_\alpha(\xi) \hat{f}(\xi), \quad \xi \in \mathbf{R}^n, \alpha < 1,$$

where $m_\alpha(\xi) = C_\alpha(1 - |\xi|^2)^{-\alpha}$ if $|\xi| < 1$, $m_\alpha(\xi) = 0$ if $|\xi| \geq 1$ and $C_\alpha = (2\pi)^{-n/2} 2^\alpha / \Gamma(1 - \alpha)$. Γ is the gamma function. If $\alpha < 1$, then the multiplier m_α is integrable and the operator T_α is well defined. But because of the factor C_α we can extend T_α to all complex α by analytic continuation or by considering the inverse Fourier transform of m_α . That is

$$\check{m}_\alpha(x) = \int_{\mathbf{R}^n} e^{ix \cdot \xi} m_\alpha(\xi) d\xi = |x|^{-n/2+\alpha} J_{n/2-\alpha}(|x|),$$

where $J_{n/2-\alpha}$ is the Bessel function of order $n/2 - \alpha$ (see Stein & Weiss [10] or Watson [12]) and we get $T_\alpha f = \check{m}_\alpha * f$.

For $\alpha = 1$ this becomes

$$T_1 f(x) = \check{m}_1 * f(x) = C \int_{S^{n-1}} e^{-ix \cdot \theta} \check{f}(\theta) d\theta = C(\check{f} d\theta)^\wedge(x),$$

where $d\theta$ denotes the surface measure on the unit sphere S^{n-1} . See Gel'fand & Shilov [8].

We will only consider the case $\text{Re } \alpha \leq (n + 1)/2$, since \check{m}_α is unbounded if $\text{Re } \alpha > (n + 1)/2$. The (L^p, L^q) estimates for T_α , $p = q$, $\alpha \leq 0$, are treated in [2], [3], [5]–[7]. The case $p < q$, $\alpha \leq 0$, $n = 2$ is easily derived from the case $p = q$ (see also Remark 2).

The origin of the techniques used in this paper and background to the subject can be found in Bochner [1], Stein [9] and Stein & Weiss [10].

To make the description of the type set of T_α easier, we define some parameters.