

# *Monotonicity Properties of Variational Integrals, $A_p$ Weights and Unique Continuation*

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**1. Introduction and statement of the results.** Let  $\Omega$  be a connected open subset of  $\mathbf{R}^n$ ,  $n \geq 3$ , and let  $A(x)$  be a symmetric  $n \times n$  matrix-valued function on  $\Omega$  on which we make the following assumptions:

(i) there exists a  $\Gamma > 0$  such that for every  $x, y \in \Omega$

$$(1.1) \quad |a_{ij}(x) - a_{ij}(y)| \leq \Gamma|x - y|, \quad i, j = 1, \dots, n;$$

(ii) there exists a  $\lambda \in (0, 1)$  such that for every  $x \in \Omega$  and  $\xi \in \mathbf{R}^n$

$$(1.2) \quad \lambda|\xi|^2 \leq \langle A(x)\xi, \xi \rangle \leq \lambda^{-1}|\xi|^2.$$

We will consider solutions,  $u$ , to the equation

$$(1.3) \quad Lu = \operatorname{div}(A(x)\nabla u(x)) = 0 \quad \text{in } \Omega.$$

By this we will mean that  $u \in H_{\text{loc}}^{1,2}(\Omega)$  and that for every  $\phi \in H_0^{1,2}(\Omega)$

$$\int_{\Omega} \langle A(x)\nabla u(x), \nabla \phi(x) \rangle dx = 0.$$

Under the assumptions (1.1) and (1.2) it is well known (see, e.g., [GT], Theorem 8.8) that every solution to (1.3) is in  $H_{\text{loc}}^{2,2}(\Omega)$ . Since all the results in this paper are of a local nature, we will assume henceforth that  $\Omega$  strictly contains  $\bar{B}_2$ , the closure of the ball with radius 2 and center at the origin.

One of the main results of this paper can be stated as follows.

**Theorem 1.1.** *Let  $u \in H_{\text{loc}}^{1,2}(\Omega)$  be a solution to (1.3). (i) If  $u \not\equiv 0$  there exist a  $p > 1$  and a constant  $A > 0$  such that for any ball  $B_R$ , such that the concentric ball  $B_{2R} \subset B_1$ , we have*

$$(1.4) \quad \left( \frac{1}{|B_R|} \int_{B_R} |u| dx \right) \left( \frac{1}{|B_R|} \int_{B_R} |u|^{-1/(p-1)} dx \right)^{p-1} \leq A.$$

*In (1.4)  $A$  and  $p$  depend on  $u$ ,  $\Gamma$ ,  $\lambda$ ,  $n$ , but do not depend on  $B_R$ . (ii) If  $u \not\equiv \text{const.}$  there exist a  $q > 1$  and a constant  $B > 0$  such that for any ball  $B_R$  as in (i)*