

Weighted Norm Inequalities for General Maximal Operators

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1. Let μ, ν be two measures on \mathbf{R}^n , \mathcal{H} a basis of \mathbf{R}^n (Section 2), and let there be associated with each $R \in \mathcal{H}$ a function ϕ_R supported in R . We consider the maximal operator $Mf(x) = \sup \int f \phi_R d\nu$, where the sup is extended over all $R \in \mathcal{H}$ with $x \in R$. For a function $g: \mathbf{R}^n \rightarrow \mathbf{R}$ and σ a measure on \mathbf{R}^n , we let $g_\sigma^*(t) = \inf\{y: \lambda_g(y) \leq t\}$, $t > 0$, where $\lambda_g(y) = \sigma\{x: |g(x)| > y\}$, $y > 0$. This is the non-increasing rearrangement of g with respect to the measure σ . We obtain (Theorem 1) the rearrangement inequality

$$(Mf)_\mu^*(\xi) \leq C \int_0^\infty \Phi(t) \phi(t) f_\nu^*(\xi t) dt,$$

where Φ is a non-increasing function given in terms of μ, ν, ϕ_R , and $\phi(t)$ is a non-increasing function which measures the (μ, ν) -overlap of \mathcal{H} . This is an extension of the inequality in [10] where \mathcal{H} was the collection of cubes in \mathbf{R}^n , and $\phi(t)$ is bounded. This inequality as well as the notion of a (μ, ν) -overlap function will give us extensions of the weighted norm inequalities of the ordinary Hardy-Littlewood maximal operator due to Muckenhoupt [12].

In Section 5 we will study in detail (μ, ν) -overlap functions for \mathcal{H} and characterize their magnitude in terms of the restricted weak type behavior of certain maximal operators. In the final section we will give some examples and indicate how some of the known results follow from our theorems.

2. By a basis $\mathcal{H} = \{R\}$ of \mathbf{R}^n we mean a collection of Borel sets such that (i) $\bigcup_{R \in \mathcal{H}} R = \mathbf{R}^n$, and (ii) \mathcal{H} has the Lindelöf property, i.e., if $\mathcal{H}' \subset \mathcal{H}$, then there is $\{R_j\}_{j \geq 1} \subset \mathcal{H}'$ with $\bigcup_{j \geq 1} R_j = \bigcup_{R \in \mathcal{H}'} R$.

For the rearrangement inequality we need the notion of an overlap function.

Definition. Given (\mathcal{H}, μ, ν) , where \mathcal{H} is a basis of \mathbf{R}^n and $\mu, \nu \geq 0$ are Borel measures on \mathbf{R}^n , finite on compact sets. A non-increasing function $\phi: (0, \infty) \rightarrow [0, \infty)$ is a (μ, ν) -overlap function for \mathcal{H} if and only if there exists $1 < B < \infty$ such that for any finite subcollection \mathcal{F} of \mathcal{H} there is $\mathcal{F}' \subset \mathcal{F}$ so that

(i) $\mu(\bigcup \mathcal{F}') \leq B \sum_{R \in \mathcal{F}'} \mu(R)$, and