

Decomposition of Banach Space Operators

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It is a long standing open question whether every Riesz operator splits as the sum of a quasinilpotent operator and a compact one (cf. [3,p.83]). It is only known, in general, for operators on a Hilbert space [12]. It is also known for arbitrary spaces under rather stringent conditions [8]. We prove in the paper that the decomposition is valid in a large class of Banach spaces including c_0 and ℓ^p , $1 \leq p < \infty$. Furthermore, we extend Stampfli's theorem [11] to these spaces—namely, for any operator T , there is a compact operator K so that $\sigma(T + K)$ is minimal. Finally, in the fourth section, we show how one can adapt Apostol's correction of singular behavior [1] to this setting.

We would like to thank Constantin Apostol and T. Alastair Gillespie for helpful discussions, and to William B. Johnson for pointing out an error in the first version of this article.

1. Preliminaries. If \mathcal{X} is a Banach space, $\mathcal{B}(\mathcal{X})$ will denote the algebra of bounded linear operators acting on \mathcal{X} , and $\mathcal{K}(\mathcal{X})$ will denote the closed two-sided ideal of compact operators. The spectrum of an operator T will be denoted by $\sigma(T)$; $\sigma_0(T)$ denotes the isolated eigenvalues of finite multiplicity, known as normal eigenvalues. The essential spectrum, $\sigma_e(T)$, is the spectrum of $\pi(T)$, the image of T in the quotient Banach algebra $\mathcal{B}(\mathcal{X})/\mathcal{K}(\mathcal{X})$. An operator is called a Riesz operator if $\sigma_e(T) = \{0\}$. Also,

$$\sigma_{\varepsilon e}(T) = \{\lambda : \text{for each } \varepsilon > 0, \|\lambda - T\|x\| < \varepsilon\|x\| \text{ for all } x \text{ in some infinite dimensional subspace } \mathcal{M}_\varepsilon\}.$$

(This always includes the boundary, $\partial\sigma_e(T)$, of $\sigma_e(T)$.)

An operator is semi-Fredholm if T has closed range and at least one of $\text{nul}(T)$, the dimension of $\ker(T)$, and $\text{nul}(T^*)$, the dimension of $\ker(T^*)$ or $\mathcal{X}/T\mathcal{X}$ is finite. The index of T is defined by

$$\text{ind}(T) = \text{nul}(T) - \text{nul}(T^*).$$

The set of semi-Fredholm operators is open, and index is a locally constant function invariant under compact perturbations. The set

$$\rho_{s-F}(T) = \{\lambda : \lambda - T \text{ is semi-Fredholm}\}$$