

Commutators in Ideals of Trace Class Operators

JOEL ANDERSON & L. N. VASERSTEIN

§1. Introduction. Fix $p > 0$ and let \mathcal{C}_p denote the Schatten p -class of operators. Thus \mathcal{C}_p stands for the familiar 2-sided ideal of operators in the ring $\mathcal{B}(\mathcal{H})$ of bounded linear operators acting on the (complex, separable, and infinite-dimensional) Hilbert space \mathcal{H} . This paper arose from an attempt to calculate $K_1(\mathcal{C}_p, \mathcal{B}(\mathcal{H}))$, the algebraic K_1 group of \mathcal{C}_p relative to the ring $\mathcal{B}(\mathcal{H})$. Due to the work of Brown and Schochet [1] and Karoubi [3], it is known that for $p > 1$ this group is 0. If $p \leq 1$, however, the question is apparently still open; moreover, the situation here is quite different. For $p \leq 1$, the determinant \det gives a homomorphism

$$\det_*: K_1(\mathcal{C}_p, \mathcal{B}(\mathcal{H})) \rightarrow \mathbb{C}^*$$

onto the multiplicative group of nonzero complex numbers. Hence for each $p \leq 1$ the relative K_1 group is not zero.

In terms of K -theory the main result to be presented here is that if $p = 1$, then the determinant is not a complete invariant. In fact, we shall see that the kernel of \det_* is isomorphic to the additive group of a vector space of uncountable dimension. Since \mathcal{C}_p is isomorphic to the n by n matrices with entries from \mathcal{C}_p and the stable range of this ring is 1 [7, 9], it follows that $K_1(\mathcal{C}_p, \mathcal{B}(\mathcal{H}))$ may be identified with the quotient of the group of invertible elements of $\mathcal{B}(\mathcal{H})$ of the form $1 + J$ where J is in \mathcal{C}_p by the commutator subgroup

$$\{1 + \mathcal{C}_p, \mathcal{B}(\mathcal{H})\}$$

generated by operators of the form

$$\{1 + J, S\} := (1 + J)S(1 + J)^{-1}S^{-1}$$

where S and $1 + J$ are invertible and J is in \mathcal{C}_p . We shall show that membership in

$$\{1 + \mathcal{C}_p, \mathcal{B}(\mathcal{H})\}$$