

Flat Chains with Positive Densities

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Introduction. Each such chain is a current representable as a countable sum of oriented m dimensional submanifolds (of differentiability class 1) multiplied by real valued density functions. The set of all these chains has interesting closure, compactness and approximation properties. Hausdorff measure is lowersemicontinuous with respect to flat convergence in this set. Isoperimetric inequalities hold. The subject was recently studied by Almgren in [1]. Here we use quite different methods, which also yield somewhat sharper results.

1. Basic concepts. We use the terminology of [2]. How a Radon measure μ over \mathbf{R}^n is related to the m dimensional Hausdorff measure \mathcal{H}^m can be described by means of the Borel regular measure \mathcal{H}_μ^m such that, for any $S \subset \mathbf{R}^n$,

$$\mathcal{H}_\mu^m(S) = \inf\{\mathcal{H}^m(A) : A \subset \mathbf{R}^n \text{ and } \mu(S \sim A) = 0\}.$$

From [2: §2.10.19(3)] we see that $E = \{x : \Theta^{*m}(\mu, x) > 0\}$ is a countably \mathcal{H}^m measurable set and $\mathcal{H}_\mu^m \geq \mathcal{H}^m \lfloor E$; in fact

$$\mu(B) \geq t\mathcal{H}^m(B) \text{ if } B \subset \{x : \Theta^{*m}(\mu, x) > t\}.$$

On the other hand [2: §2.10.19(1)] assures us that

$$\mu(B) = 0 \text{ if } B \subset \mathbf{R}^n \sim E \text{ and } \mathcal{H}^m(B) < \infty.$$

Consequently the following three conditions are equivalent:

- (I) $\Theta^{*m}(\mu, x) > 0$ for μ almost all x .
- (II) $\mathcal{H}_\mu^m = \mathcal{H}^m \lfloor \{x : \Theta^{*m}(\mu, x) > 0\}$.
- (III) \mathbf{R}^n is countably \mathcal{H}_μ^m measurable.

Next we recall the structure theorem [2: §3.3.13] characterizing unrectifiability by the vanishing of the integralgeometric measure \mathcal{I}_1^m and infer: