

# *Exceptional Boundary Sets for Solutions of Elliptic Partial Differential Inequalities*

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**§1. Introduction.** Let  $L$  denote an elliptic second order linear partial differential operator (see (2.1)), with bounded coefficients defined in a domain  $\Omega$  in  $\mathbf{R}^n$ . Let  $\Gamma$  be a subset of the boundary of  $\Omega$ , with its Hausdorff dimension appropriately restricted. We prove, under suitable conditions, that if  $u$  is a  $C^2$  solution in  $\Omega$  of  $Lu \leq \text{constant}$ , with the lower bound  $u \geq \text{constant}$ , then the knowledge that  $u \geq 0$  on  $\partial\Omega/\Gamma$  near  $\Gamma$  implies that also  $u \geq 0$  on  $\Gamma$ . The allowable Hausdorff dimension of the “exceptional set”  $\Gamma$  depends on the ellipticity constants for  $L$  and on the angular openings of required external cones to  $\Omega$  with vertices on  $\Gamma$ . For example, if the principal part of  $L$  is the Laplace operator, and if at each point on  $\Gamma$  the domain  $\Omega$  lies on one side of a tangential  $(n - 1)$ -dimensional hyperplane, then the Hausdorff dimension of  $\Gamma$  need only be less than  $n - 1$ .

These results extend standard versions of so-called “Phragmén–Lindelöf principles”. Phragmén–Lindelöf principles generally describe the behavior of a solution of a differential equation or inequality when the solution is known to be nonnegative on the boundary of a domain except at an exceptional boundary point. Under appropriate growth restrictions on the solution, one concludes that the solution must be nonnegative also at the exceptional boundary point.

For elliptic equations Phragmén–Lindelöf principles were first proved by Gilbarg [5] in two dimensions, and by Hopf [9] in general  $\mathbf{R}^n$ . Further refinements and strengthening of these results were given by Serrin [17] and Friedman [4]. Miller [12, 13] obtained Phragmén–Lindelöf results for general uniformly elliptic operators with no continuity at all assumed on the coefficients. He also established existence of maximal and minimal Phragmén–Lindelöf indices for classes of uniformly elliptic operators with eigenvalues of the matrix of second order coefficients contained within a fixed range. Oddson [14, 15] derived explicit formulas for these indices in the two-dimensional case. For other discussions of