

Brownian Motion and Area Functions

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§0. Introduction

Let $D = \{z \in \mathbb{C}: |z| < 1\}$ be the unit disc in the complex plane and let $T = \partial D$ be the unit circle. $dm = d\vartheta/2\pi$ will denote the normalized Lebesgue measure on T . For every $0 < \alpha < 1$ we define the Stoltz domain $\Gamma_\alpha(\vartheta)$ to be the interior of the smallest convex set containing the disc $\{|z| < \alpha\}$ and the point $e^{i\vartheta}$. If f is an integrable function defined on T , we will write u for its harmonic extension to D . Define

$$(0.1) \quad A_\alpha(f)(\vartheta) = \left(\int_{\Gamma_\alpha(\vartheta)} |\nabla u(z)|^2 dx dy \right)^{1/2}$$

and

$$(0.2) \quad g_*(f)(\vartheta) = \left(\frac{1}{\pi} \int_D \log \frac{1}{|z|} P_\vartheta(z) |\nabla u(z)|^2 dx dy \right)^{1/2},$$

where $z = x + iy$ and

$$P_\vartheta(z) = \frac{1 - |z|^2}{|z - e^{i\vartheta}|^2},$$

the Poisson kernel for D . $A_\alpha(f)$ is called the Lusin area function and $g_*(f)$ is pointwise comparable with the classical Littlewood–Paley g_* -function.

It is not difficult to show that if $A_\alpha(f)$ is a bounded function, then f belongs to the space BMO and hence the John–Nirenberg theorem implies that f is in the exponential class. In other words, $\exp(C|f|) \in L^1(T)$ for some positive constant C . (See [9] for definition and properties of BMO-functions.) But in fact more is true. It was proved in [5] that if $A_\alpha(f)(\vartheta) \leq 1$ a.e., then there are constants C_α and C'_α depending only on α such that

$$(0.3) \quad \int_T \exp(C_\alpha |f - f_0|^2) dm \leq C'_\alpha,$$

where $f_0 = \int_T f dm$. It is easy to show that $A_\alpha(f)(\vartheta) \leq C''_\alpha g_*(f)(\vartheta)$ a.e. for some constant C''_α . Thus, the boundedness of $g_*(f)$ also gives exponential