

# *A General Variational Identity*

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In 1965 Pohožaev discovered an important identity for solutions of the Dirichlet problem

$$(1) \quad \Delta u + f(u) = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

This identity enabled him to show that problem (1) has no nontrivial solutions when  $\Omega$  is a bounded star-shaped domain in  $\mathbf{R}^n$  and  $f = f(u)$  is a continuous function on  $\mathbf{R}$  satisfying the condition

$$(n-2)uf(u) - 2nF(u) > 0 \quad \text{for } u \neq 0,$$

where  $F$  denotes the primitive  $F(u) = \int_0^u f(t) dt$  of  $f$ .

Here we shall obtain a corresponding identity for extremals of the general variational problem

$$(2) \quad \delta \int_{\Omega} \mathcal{F}(x, u, Du) dx = 0,$$

see formula (4) below. This result can be applied similarly to demonstrate that no nontrivial extremals  $u$  can exist in a star-shaped domain  $\Omega$  when  $u = 0$  on  $\partial\Omega$  and the function  $\mathcal{F} = \mathcal{F}(x, u, p)$  satisfies appropriate conditions (see Theorem 1). For the special case  $\mathcal{F}(x, u, p) = G(p) - F(x, u)$  the results are considerably easier to state, and a number of immediately ensuing applications are given in §2.

In §3 we apply our identity to an eigenvalue problem, generalizing a result noted in [1]. The case of unbounded domains is treated in §4. Here two essentially different situations arise, the first when  $\Omega = \mathbf{R}^n$ , the second when  $\Omega$  has the structure of a “half-space”; cf. Esteban and Lions [2].

The existence of a general identity for the variational problem (2) leads one to expect a similar identity for vector-valued extremals, mapping  $\Omega$  into  $\mathbf{R}^N$ , and for higher order problems. This expectation indeed can be realized, as we show in the final section. The conclusions are new even for the system

$$\Delta u^i + f_i(u) = 0, \quad i = 1, \dots, N,$$