

Removable Sets for Weak L^p

ROBERT KAUFMAN

§1. Let L_{p^∞} , or *weak L^p* , be the class of measurable functions f on R^2 such that $\sup \lambda^p m(|f| > \lambda) < +\infty$. We are interested in removable sets for analytic functions of class L_{p^∞} ($2 < p < +\infty$); more precisely in this problem: f is in $L_{p^\infty}(R^2)$ and analytic outside a closed set E . It is known that E has finite Hausdorff h -measure; when does it follow that f is equal almost everywhere to an entire function? The answer is given in Theorems 1 and 2.

The spaces L_{p^∞} have an ancillary role and it is worthwhile to explain the interest in L_{p^∞} instead of more usual spaces. In fact, the corresponding problem for L^p (also called L_{pp} or *strong L^p*) is relatively trivial. In proving the sufficiency of the condition on h , we are guided by knowledge of the function space dual to L_{p^∞} ; in proving the necessity of the same condition, an entirely different method is used, with no apparent relation to the sufficiency, or indeed to the known techniques in constructions of examples, [1].

Finally, the space L_{p^∞} arises naturally in the theory of Hausdorff measures; if $\alpha = (p-2)(p-1)^{-1}$, $0 < \alpha < 1$, and a closed set E has positive Λ^α -measure, then by a theorem of Frostman [4] and elementary estimations, E is *not* removable for L_{p^∞} . For certain sets E of positive Λ^α -measure, the function f (analytic off E but not entire) cannot obey the condition $m(|f| > \lambda) = o(\lambda^{-p})$, $\lambda \rightarrow +\infty$, [8, p. 109]. These two facts suggest a simple answer to our main problem, which turns out to have a much more subtle solution.

§2. Theorem 1. *Let h be increasing and positive on $(0, +\infty)$ and suppose that $\sum_0^\infty [h(2^{-m})2^{\alpha m}]^{-1/1-\alpha} < +\infty$. Let E be a closed set of finite Hausdorff h -measure.*

If f is in $L_{p^\infty}(R^2)$, and is analytic off E , then f can be modified (on E) to be an entire function.

After proving Theorem 1, we obtain a converse to it, and then explain some relations to the theory of capacities based on Lebesgue spaces. Our converse to Theorem 1 is not the strongest one that might be conjectured; some indications about other possible results are discussed.

Proof of Theorem 1. Let φ be a C^1 function of compact support; we prove that $\int \bar{\partial} \varphi f(z) dx dy = 0$, where $\bar{\partial}$ is defined by $2\bar{\partial} = \partial/\partial x + i\partial/\partial y$. Then