

# *Kinematic and Tube Formulas for Piecewise Linear Spaces*

J. CHEEGER, W. MÜLLER & R. SCHRADER

*Dedicated to the memory of P. Wintgen*

In a recent article [CMS1] we showed that the Lipschitz–Killing curvatures of a piecewise linear (p.l.) space are close (in the sense of measures) to those of a smooth space, if the p.l. space approximates the smooth space suitably. The definitions of the mean curvature and the scalar curvature of a p.l. space are due to Steiner[St] and Regge [R] respectively. Higher-order Lipschitz–Killing curvatures appear in the asymptotics of the heat kernel on a p.l. space [C].

Physicists are interested in these curvatures (and speak of Regge calculus) in the context of attempts to quantize gravity (see e.g. [CMS1], [CMS2] and the references given there, which also contain applications in classical general relativity; furthermore, see also [FFLR], [FL], [L]).

It is the purpose of this article to show that Lipschitz–Killing curvatures of p.l. spaces also appear in other contexts, namely in Chern’s kinematic formula and Weyl’s tube formula. The observation concerning the kinematic formula was first made by P. Wintgen, who announced a proof of it in [Wi], based on his notion of the “normal cycle.” His sudden death prevented a publication. The proof of the kinematical formula we present here is essentially due to him, and our proof of the tube formula also relies on the normal cycle.

We begin the discussion by recalling Chern’s kinematic formula in the smooth case. Let  $X^n \subset E^N$  be a compact Riemannian manifold  $X^n$  isometrically embedded in some euclidean space  $E^N$ . Consider the affine Grassmannian  $G_{N,k}$  of all  $k$ -planes  $E^k \subset E^N$ . Let  $dE^k$  be the natural measure on  $G_{N,k}$  which is invariant under the euclidean group of motions. In integral geometry (see e.g. [S], [SW] for an overview) one introduces the quantities

$$(1) \quad M_k(X^n) = \frac{V_N}{V_{N-k}\gamma_{N,k}} \int_{G_{N,k}} \chi(X^n \cap E^k) dE^k, \quad k = 0, \dots, N.$$