

On the Uniqueness of Capillary Surfaces without Gravity over an Infinite Strip

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§1. Let Ω be a domain in \mathbf{R}^n . Consider the following boundary value problem:

$$(1) \quad \begin{cases} \operatorname{div} Tu = H & \text{in } \Omega, \\ Tu = \frac{Du}{\sqrt{1+|Du|^2}} \\ Tu \cdot \nu = \cos \gamma & \text{on } \partial\Omega, \end{cases}$$

where ν is the unit normal of $\partial\Omega$, H and γ are constants, $\frac{\pi}{2} \geq \gamma \geq 0$.

If Ω is bounded, it was proved in [C-F] that the solution of (1) is unique up to an additive constant if it exists. However, this is not true in general if Ω is unbounded. In this paper we want to discuss the problem of uniqueness for the case when Ω is the infinite strip $\mathbf{R} \times (-\frac{1}{2}, \frac{1}{2})$ in \mathbf{R}^2 .

In this case, it is obvious that there are infinite many solutions, no two of which differ by a constant. In fact, if (1) has a solution, then $H = 2\cos\gamma$ (Lemma 1 below). Hence, it is easy to see that if $\gamma = \frac{\pi}{2}$, there are infinitely many linear solutions. If $\frac{\pi}{2} > \gamma \geq 0$, define the cylinder

$$(2) \quad \Phi(x) = \Phi(x^1, x^2) = -\sqrt{\left(\frac{1}{2\cos\gamma}\right)^2 - (x^2)^2}, \quad x = (x^1, x^2) \in \Omega.$$

Φ is a solution of (1), so is any suitable rigid rotation of $\Phi + \text{constant}$, that is any function of the form

$$\Phi_\beta(x^1, x^2) = \frac{1}{\alpha} \left[\Phi\left(\frac{1}{\alpha}x^1, x^2\right) + \beta x^1 \right] + \text{constant},$$

where α, β are constants, $\alpha > 0$ and $\alpha^2 + \beta^2 = 1$.

However, we shall prove that these are the only solutions if $\frac{\pi}{2} \geq \gamma > 0$. Note that we do not assume any growth condition at infinity for the solution.