

Analytic and Univalent Functions with Integral Representations Involving Complex Measures

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§1. Introduction. Let U denote the set of functions f that are analytic and univalent in $\Delta = \{z: |z| < 1\}$, and let S denote the subset of U where $f(0) = 0$ and $f'(0) = 1$. Also, let \mathcal{F} denote the set of functions f for which

$$(1) \quad f(z) = \int_{\partial\Delta} \frac{1}{(1-xz)^2} d\mu(x)$$

for $|z| < 1$ and $\mu \in \Lambda$, where Λ denotes the set of (finite) complex-valued measures on $\partial\Delta$.

This work was motivated by the question, Is $U \subset \mathcal{F}$? D. R. Wilken probably first asked such a question over 15 years ago. This was after L. Brickman, D. R. Wilken and the author showed that each starlike function in S has the representation

$$(2) \quad f(z) = \int_{\partial\Delta} \frac{z}{(1-xz)^2} d\mu(x)$$

for $|z| < 1$, where $\mu \in \Lambda^*$ and Λ^* denotes the set of probability measures on $\partial\Delta$ [6]. There are functions in S which do not have the representation (2) with $\mu \in \Lambda^*$. This is a consequence of a more general fact about the closed convex hull of close-to-convex mappings (see [6], p. 104).

The problem of whether $U \in \mathcal{F}$ has attracted more attention in the last few years. One result of A. Baernstein will be mentioned later. P. Bourdon and J. A. Cima consider this problem in [3], where the emphasis is on finding sufficient conditions on an analytic function so that it has a Cauchy–Stieljes representation

$$(3) \quad g(z) = \int_{\partial\Delta} \frac{1}{1-xz} d\mu(x)$$

for $|z| < 1$, where $\mu \in \Lambda$ (also see [1]).