

Existence and Asymptotic Behavior for Strong Solutions of the Navier–Stokes Equations in the Whole Space

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We shall consider the initial value problem for the nonstationary Navier–Stokes equations in the whole space, namely,

$$(0.1) \quad \begin{cases} v' - \mu \Delta v + (v \cdot \nabla)v = f - \nabla p, & \text{in }]0, T[\times \mathbf{R}^n, \\ \nabla \cdot v = 0, & \text{in }]0, T[\times \mathbf{R}^n, \\ v = a(x), & \text{in } \mathbf{R}^n, \\ \lim_{|x| \rightarrow +\infty} v(t, x) = 0, & \text{for } t \in]0, T[, \end{cases}$$

where $T \in]0, +\infty]$, μ is a positive constant, $v' = \partial v / \partial t$, and

$$((v \cdot \nabla)v)_j = \sum_{i=1}^n v_i \frac{\partial v_j}{\partial x_i}, \quad j = 1, \dots, n.$$

The vector field $v(t, x)$ and the scalar field $p(t, x)$ are unknowns. The initial velocity $a(x)$ and the external forces $f(t, x)$ are given. The pressure is determined by the condition $\lim_{|x| \rightarrow +\infty} p(t, x) = 0$, as $|x| \rightarrow +\infty$. Moreover,

$$(0.2) \quad \nabla \cdot f = 0 \text{ a. e. in }]0, T[\quad \text{and} \quad \nabla \cdot a = 0.$$

The first condition (0.2) is not strictly necessary.

Our main concern will be the asymptotic behaviour of the solutions, and the core of the paper are the a priori estimates in §§1 and 3. Appendices and proofs concerning the existence of the solutions (estimates of §1 apart) are presented mainly for the sake of completeness. The reader acquainted with Navier–Stokes equations should skip §2 and appendices, or do them by different methods. It is worth noting that some technical difficulties can be avoided by assuming more regularity on f (as for instance, by assuming that $f \in L^2(0, T; L^\alpha)$, or that $f \in L^\infty(0, T; L^\alpha)$, instead of (1.1)).

By a solution of problem (0.1) we mean a divergence free vector field $v(t, x) \in L^1(0, T; L^2_{\text{loc}})$ such that

$$(0.1') \quad \int_0^T \int [v \cdot \varphi' + \mu v \cdot \Delta \varphi + (v \cdot \nabla)\varphi \cdot v + f \cdot \varphi] dx dt = - \int a \varphi|_{t=0} dx,$$