

# Classification of Quadratic Harmonic Maps of $S^3$ into Spheres

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**§1. Introduction and statement of the result.** In 1972 R. T. Smith posed the problem of classifying all harmonic maps  $f: S^m \rightarrow S^n$ ,  $m \geq 2$ , between Euclidean spheres whose components are homogeneous (harmonic) polynomials of (fixed) degree  $k$  (cf. [2, 6, 7] and [3], Problem (4.4), p. 70). Full linear ( $k = 1$ ) harmonic maps are isometries of  $S^m$  and, by Calabi's rigidity theorem, any full  $k$ -homogeneous harmonic map of  $S^2$  is a standard minimal immersion  $f_{\lambda_k}: S^2 \rightarrow S^{2k}$  [1, 8, 9]. The object of this paper is to give a classification of full  $k$ -homogeneous harmonic maps in the first (nonrigid) range of  $m$  and  $k$  not covered by the results above, i.e., when  $m = 3$  and  $k = 2$ .

**Classification Theorem.** *Full quadratic ( $k = 2$ ) harmonic maps of  $S^3$  into  $S^n$  exist only if  $2 \leq n \leq 8$  and  $n \neq 3$ . Moreover, if  $f: S^3 \rightarrow S^n$  is such a map, then there exist  $U \in O(4)$ ,  $V \in O(n+1)$  and a symmetric positive definite matrix  $B \in S^2(\mathbb{R}^{n+1})$  such that*

$$V \circ f \circ U = B \circ f_n,$$

where  $f_n: S^3 \rightarrow S^n$  defined by

$$f_n(x, y, u, v) = \begin{cases} (x^2 + y^2 - u^2 - v^2, 2(xu - yv), 2(xv + yu)), & [\text{Hopf map}] & n = 2 \\ (x^2 + y^2 - u^2 - v^2, 2xu, 2xv, 2yu, 2yv), & & n = 4 \\ (x^2 - y^2, u^2 - v^2, 2xy, \sqrt{2}(xu + yv), \sqrt{2}(yu - xv), 2uv), & & n = 5 \\ (\frac{1}{\sqrt{2}}(x^2 + y^2 - u^2 - v^2), \frac{1}{\sqrt{2}}(x^2 - y^2), \frac{1}{\sqrt{2}}(u^2 - v^2), \\ \quad \sqrt{2}xy, \sqrt{3}(xu + yv), \sqrt{3}(yu - xv), \sqrt{2}uv), & & n = 6 \\ (x^2 - y^2, u^2 - v^2, 2xy, \sqrt{2}xu, \sqrt{2}xv, \sqrt{2}yu, \sqrt{2}yv, 2uv), & & n = 7 \\ f_{\lambda_2}(x, y, u, v), [f_{\lambda_2} = \text{a standard minimal immersion}] & & n = 8 \end{cases}$$

For fixed  $n$ , the matrices  $B$  (corresponding to the various maps  $f$ ) form an (open) convex body  $I_f$  lying in a finite-dimensional vector space. Finally,  $I_{f_2} = \text{point}$ ,  $I_{f_4} = \text{segment}$ ,  $I_{f_5} = 2\text{-disk}$ ,  $I_{f_6} = (\text{finite}) \text{ solid cone}$ ,  $\dim I_{f_7} = 5$  and  $\dim I_{f_8} = 10$ .