

Module Maps and Hochschild–Johnson Cohomology

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§1. Introduction. Let us suppose that R is a von Neumann algebra on a Hilbert space H . We may regard the algebra $\mathcal{B}(H)$ of all operators on H as both an R bimodule and an R' bimodule. In an unpublished article [6], Uffe Haagerup proved that any normal completely bounded R' bimodule map $\Phi: \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ has the form

$$\Phi(b) = \sum x_k b y_k \quad (b \in \mathcal{B}(H), x_k, y_k \in R).$$

In this paper we begin by showing that there is an analogous decomposition for non-normal maps. Using the suggestive notation of [5], we show that the completely bounded R' bimodule maps $\Phi: \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ are just those of the form

$$\Phi(b) = \int x b y \, dm(x, y),$$

where m is an appropriate “mean” on $R \otimes R$. To be more precise, given $x, y \in R$, let $\psi_0(x \otimes y): \mathcal{B}(H) \rightarrow \mathcal{B}(H)$ be the completely bounded R' bimodule map defined by

$$\psi_0(x \otimes y)(b) = x b y.$$

This determines a map ψ_0 from $R \otimes R$ into the R bimodule $\mathcal{M}_{R'}$ of completely bounded R' bimodule maps on $\mathcal{B}(H)$. Slightly modifying a notion of [5], we show that ψ_0 has a unique weak* continuous extension to an isometric weak* homeomorphic R bimodule isomorphism $\psi: R \hat{\otimes}_h^\sigma R \simeq \mathcal{M}_{R'}$, where $R \hat{\otimes}_h^\sigma R$ is a natural, intrinsically defined completion of $R \otimes R$. This isomorphism has several important consequences. To begin with, the argument shows that if all of the normal states on R are vector states, then any bounded R' module map $\mathcal{B}(H) \rightarrow \mathcal{B}(H)$ is automatically completely bounded (Haagerup had proved this for normal maps). On the other hand, under this isomorphism, the quasi-expectations of Bunce and Paschke correspond exactly to the “reduced virtual diagonals” in the tensor product. This enables one to give an elementary proof of the equivalence of σ -amenability, injectivity, and the existence of a normal virtual diagonal for von Neumann algebras. In contrast with the earlier treatments of this result [3], [1], [5], one no longer has to appeal to Connes’ famous result that injectivity is