

On the Majorization of Analytic Functions

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§1. Introduction and results

1.1. Let G be an open set in the plane, and let f be a function analytic in G and continuous in \bar{G} , the closure of G . We say that a function μ defined for $t \geq 0$ is a *majorant* if μ is nonnegative, nondecreasing and continuous, and

$$(1.1) \quad \mu(2t) \leq 2\mu(t)$$

for $t \geq 0$. We consider the following majorization problem for analytic functions. Suppose that

$$(1.2) \quad |f(z_1) - f(z_2)| \leq \mu(|z_1 - z_2|)$$

for all z_1 and z_2 on ∂G , the boundary of G . What kind of upper bound can be obtained for $|f(z_1) - f(z_2)|$ when $z_1, z_2 \in \bar{G}$?

Suppose that G has at least two finite boundary points. Gehring, Hayman and the author [3, Theorem 1] showed that if $\mu(t) = Mt^\alpha$ where $M > 0$ and $0 \leq \alpha \leq 1$, then (1.2) for all $z_1, z_2 \in \partial G$ implies that (1.2) holds for all $z_1, z_2 \in \bar{G}$. Tamrazov [10] and the author [5, Theorem 1] showed more generally that if $\log \mu(e^t)$ is a concave function of t , then (1.2) on ∂G implies (1.2) in \bar{G} . Tamrazov [9, Theorem 9.3, p. 167] had shown that the concavity condition is necessary for this result to be true. In all these cases one must assume that in any unbounded component of G , the function f satisfies a suitable growth restriction such as $f(z) = o(|z|)$ as $z \rightarrow \infty$. For details we refer to [3, 5, 10].

1.2. In general, the best result one can hope for is that

$$(1.3) \quad |f(z_1) - f(z_2)| \leq C\mu(|z_1 - z_2|)$$

for $z_1, z_2 \in \bar{G}$ for some constant $C > 1$. Examples of Rubel, Shields and Taylor [6, p. 34 and Proposition 4.3, p. 38] show that one sometimes needs to take $C > 1$, even if G is the unit disk. So given an open set G , one can ask if there is an absolute constant C or a constant C depending on G only such that (1.2) on ∂G implies (1.3) in \bar{G} for any f and μ . We assume from now on that each