

Lipschitz Continuity of Solutions and Interfaces of the N -dimensional Porous Medium Equation

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Introduction. We consider the Cauchy problem for the porous medium equation

$$(0.1) \quad u_t = \Delta u^m, \quad x \in \mathbf{R}^N, \quad t > 0,$$

where $m > 1$ is a constant, with initial data

$$(0.2) \quad u(x, 0) = u_0(x), \quad x \in \mathbf{R}^N.$$

We suppose that u_0 is a nonnegative, integrable function in \mathbf{R}^N that does not vanish identically and whose support is contained in the ball $B = B(0, 1)$ of center 0 and radius 1.

Equation (0.1) appears in a number of applications to describe the evolution of diffusion processes, in particular the flow of a gas in a porous medium, cf. [M] or [A1]. There u stands for the density, $v = \frac{m}{m-1} u^{m-1}$ is the pressure and $-\nabla v = -(v_{x_1}, \dots, v_{x_N})$ is the local velocity of the gas. We remark that v satisfies the equation

$$(0.3) \quad v_t = (m-1)v\Delta v + |\nabla v|^2.$$

Since equation (0.1) (or (0.3)) is degenerate at the points where $u = v = 0$, in general there are no classical solutions of (0.1), (0.2). Nevertheless there exists a unique weak solution $u = u(x, t)$ of the Cauchy problem above satisfying

- (i) $u \in C([0, \infty); L^1(\mathbf{R}^N)) \cap L^\infty((\delta, \infty) \times \mathbf{R}^N)$ for every $\delta > 0$, and $u(x, t) \geq 0$ for every $(x, t) \in Q = \mathbf{R}^N \times (0, \infty)$,
- (ii) $u_t - \Delta(u^m) = 0$ in $\underline{D}'(Q)$,
- (iii) $u(0) = u_0$,
- (iv) u is Hölder continuous in Q and $u \in C^\infty(\Omega)$, where Ω is the positivity set of u , $\Omega = \{(x, t) \in Q; u(x, t) > 0\}$.