

Content and Harmonic Measure: An Extension of Hall's Lemma

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One version of Hall's Lemma states that if $D = \{|x| < 1\}$ in \mathbf{R}^m , $P = (0, 0, \dots, 0)$, E is a closed set in $\{\frac{1}{2} \leq |x| < 1\}$ and E^* is the radial projection of E onto $|x| = 1$, then

$$\omega^P(E, D \setminus E) \geq c\Lambda_{m-1}(E^*)$$

where ω is the harmonic measure, Λ_{m-1} is the $m-1$ -dimensional Hausdorff measure and $c > 0$ is a constant depending only on m . Other versions involve other types of projections in balls or half spaces.

One purpose of this paper is to extend Hall's Lemma to domains where the natural projections are not available. We shall prove the following.

Theorem 1. *Suppose that D is a domain in \mathbf{R}^m , $m \geq 2$, satisfying properties (A), (B), and (C) below, and P is a fixed point in D . Then there exist positive constants c and α depending on D and P only so that for any relatively closed subset E of D ,*

$$(0.1) \quad \omega^P(E, D \setminus E) \geq cM_{m-1}(E)^\alpha,$$

where ω is the harmonic measure and M_{m-1} is the $m-1$ -dimensional content. In particular, (0.1) holds when D is a chord-arc domain ($m = 2$), a Lipschitz domain or a BMO_1 domain. When ∂D is $C^{1+\varepsilon}$, $\varepsilon > 0$, we can choose α to be 1.

The exact statements of properties (A)–(C) will be given in §2. For a set S in \mathbf{R}^m , the γ -dimensional content of S is defined to be

$$M_\gamma(E) = \inf \sum_n r_n^\gamma$$

where the infimum is taken over all coverings of E with a countable number of balls with radii r_n [3]. The content is a capacity, and Borel sets are capacitable, see [2], [3], and [7]. Let Λ_γ be the γ -dimensional Hausdorff measure. It is clear that $M_\gamma \leq \Lambda_\gamma$, and M_γ and Λ_γ have the same null sets. For sets lying on a "nice" surface, Λ_{m-1} and M_{m-1} are comparable. For sets inside a domain D and away from the boundary, M_{m-1} is more realistic than Λ_{m-1} for the