

# Wiener Criterion and Potential Estimates for the Obstacle Problem

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**Introduction.** Let  $\psi$  be an arbitrary given function, defined up to sets of Newtonian (external) capacity zero in  $\mathbf{R}^n$ ,  $n \geq 3$ , taking its values in the extended reals  $[-\infty, +\infty)$ . We shall call such a  $\psi$  an *obstacle*.

We introduce the notion of *regular point*  $x_0 \in \mathbf{R}^n$  of  $\psi$  as follows. We require that every local weak solution  $u \in H^1(\Omega)$  of the obstacle problem

$$u \geq \psi \text{ q. e. in } \Omega, \quad Lu \geq 0 \text{ in } \Omega, \quad (u - \psi)Lu = 0$$

on some bounded open neighborhood  $\Omega$  of  $x_0$  in  $\mathbf{R}^n$  be (finite and) *continuous* at  $x_0$ . By  $L$  we are denoting an arbitrary second order uniformly elliptic linear partial differential operator in divergence form, with bounded measurable coefficients in  $\Omega$ ; see §1.6.

The first main result of this paper, Theorem 5.2, shows that the regular points of  $\psi$  do not depend on the specific choice of  $L$  and can be characterized by a condition of the type of the classical Wiener's criterion of potential theory. Such a condition requires one-sided level sets of  $\psi$  in balls  $B_\rho(x_0)$  of radius  $\rho$  around  $x_0$  not to become too “thin” as  $\rho \rightarrow 0$ , when measured by their relative capacities with respect to the balls  $B_\rho$ 's—see Definition 3.1 and §5.

Our second result is a “structural” *potential estimate* for an arbitrary local weak solution  $u$  of the obstacle problem above. This estimate, given by Theorems 6.1 and 6.2, can be seen as an analogue of the energy decay's estimates related to the so-called *Saint-Venant's principle* of linear elasticity [30, 33]. In particular, we estimate the modules of continuity of  $u$  at an arbitrary regular point of  $\psi$ .

Our results will be formulated in terms of the *Wiener modulus* of  $\psi$  at  $x_0$  which we introduce in §4. This modulus describes the relevant pointwise “variational” behavior of the obstacle, independently of any a priori smoothness requirement.

An interesting feature of our estimates is their “variational stability,” in the sense, for instance, of so-called  $\Gamma$ -convergence of functionals, which will be exploited in further work; see [10].