

Manifolds of Partially Positive Curvature

H. WU

Introduction. This paper studies certain curvature functions on a Riemannian manifold which interpolate between the sectional curvature and the Ricci curvature. In the course of our investigation, we will have something new to say about the latter two well-known curvature functions as well. By definition, a Riemannian manifold M has q -positive (resp., q -nonnegative) curvature if and only if for each $x \in M$ and for any $(q+1)$ orthonormal vectors $\{e_0, e_1, \dots, e_q\}$ in M_x , $\sum_{i=1}^q K(e_0, e_i) > 0$ (resp., ≥ 0), where $K(e_0, e_i)$ denotes the sectional curvature of the plane spanned by e_0 and e_i . Thus 1-positive (1-nonnegative) curvature is equivalent to positive (nonnegative) sectional curvature, and $(n-1)$ -positive ($(n-1)$ -nonnegative) curvature is equivalent to positive (nonnegative) Ricci curvature. For convenience, we introduce a function $K_q: M \rightarrow \mathbf{R}$ by $K_q(x) =$ the minimum of all such $\sum_i K(e_0, e_i)$ in M_x as above. Note the elementary fact that if $K_q \geq 0$, then $K_{q+1} \geq K_q$, and if $K_q \leq 0$, then $K_{q+1} \leq K_q$. Now suppose M is compact with (C^∞) boundary ∂M . Let $\Lambda_q: \partial M \rightarrow \mathbf{R}$ be the function $\Lambda_q(y) =$ the minimum of all sums of any q eigenvalues of the second fundamental form S of ∂M at y with respect to the outward-pointing unit normal at y . Throughout this paper, the symbols K_q , Λ_q and S will be reserved exclusively for the quantities just defined. Leaving the detailed bibliographical discussion of the principal theorems of this paper to the end of this introduction, we shall concentrate on the mathematics for now.

Theorem 1. *Let M be an n -dimensional compact Riemannian manifold with boundary ∂M and let q be an integer satisfying $1 \leq q \leq n-1$. Suppose M has either of the following properties:*

- (i) $\Lambda_q > 0$ and $\min_{\partial M} \Lambda_q + \rho_0 \min_M K_q > 0$, where ρ_0 denotes the maximum of the distance from ∂M .
- (ii) $\Lambda_q \geq 0$, $K_q \geq 0$, and, in addition, $K_q > 0$ in a neighborhood of ∂M .

Then: