

The Fredholm Structure of an n -Multicyclic Operator

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§1. Introduction. In a series of articles, the author has analyzed the spectral structure of multicyclic and rationally multicyclic operators [13] [14] [16] (see also the joint article [17], in cooperation with L. Rodman; a large part of these results have been included in Chapters 11 and 13 of the monograph [3] [15]). A (bounded linear) operator T , acting on a complex separable infinite-dimensional Banach space \mathcal{X} , is called *n -multicyclic* if

$$(\star) \quad \mathcal{X} = \bigvee \{p_1(T)y_1, p_2(T)y_2, \dots, p_n(T)y_n : \\ p_1, p_2, \dots, p_n \text{ run over all polynomials}\}$$

for a suitable n -tuple of vectors $(y_1, y_2, \dots, y_n) \in \mathcal{X}^{(n)}$, the direct sum of n copies of \mathcal{X} . Here \bigvee denotes “the closed linear span of”, and (y_1, y_2, \dots, y_n) is called a *multicyclic n -tuple* for T . The operator T is called *rationally n -multicyclic* if there exists (y_1, y_2, \dots, y_n) in $\mathcal{X}^{(n)}$ (a rationally multicyclic n -tuple for T) such that (\star) is satisfied when p_1, p_2, \dots, p_n run over all rational functions with poles outside the spectrum $\sigma(T)$, of T .

Recall that an operator T is *semi-Fredholm* if $\text{ran } T$ is closed, and either $\ker T$ or $\frac{\mathcal{X}}{\text{ran } T}$ is finite-dimensional. In this case, the semi-Fredholm index of T is defined by

$$\text{ind } T = \dim \ker T - \dim \frac{\mathcal{X}}{\text{ran } T} = \dim \ker T - \dim \ker T^*.$$

If $\text{ind } T$ is finite, then T is called a Fredholm operator. The open set

$$\rho_{s-F}(T) = \{\lambda \in \mathbf{C} : \lambda - T \text{ is semi-Fredholm}\}$$

is the *semi-Fredholm domain* of T . (The reader is referred to [18] for the properties of semi-Fredholm operators.) If T is either n -multicyclic, or rationally n -multicyclic, then $\dim \ker(\lambda - T)^*$ cannot exceed n , and therefore $\text{ind}(\lambda - T) \geq -n$ for all $\lambda \in \rho_{s-F}(T)$ [13].