

In General a Stochastic Operator Does Not Preserve a Measure

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Let \mathcal{S} be the set of *stochastic operators*, i.e., positive contractions T on $L^1(X, \mathbf{M}, \mu)$ satisfying $T^* \mathbf{1} = \mathbf{1}$, where (X, \mathbf{M}, μ) is a Lebesgue probability space. Let \mathcal{S}_λ be the subset of \mathcal{S} consisting of those operators which preserve a σ -finite measure λ , $\lambda \sim \mu$. \mathcal{S} is endowed with the *strong operator topology*; since \mathcal{S} is closed in $\mathcal{L}(L^1, L^1)$, this makes it a Baire space. For details of the above, see Iwanik [Iw], also the book of Foguel [F] which uses a very different notation. Let \mathcal{G} be the group of positive invertible isometries on $L^1(X, \mathbf{M}, \mu)$, again with the strong operator topology; \mathcal{G} is also a Baire space. These isometries can be identified in a natural way with the invertible nonsingular transformations (automorphisms) of (X, \mathbf{M}, μ) : if τ is an automorphism, the corresponding isometry T_τ is given by $T_\tau f(x) = f(\tau^{-1}x) \frac{d\mu \circ \tau^{-1}}{d\mu}(x)$, and every positive invertible isometry arises in this way. (The strong operator topology is often called the coarse topology on the automorphisms.) We denote by \mathcal{M}_λ the (closed) subgroup of \mathcal{G} consisting of isometries (automorphisms) which preserve the σ -finite measure λ , $\lambda \sim \mu$. Iwanik [Iw] has shown that

- (a) \mathcal{S} is the closed convex hull of \mathcal{G} .
- (b) For each $\lambda \sim \mu$, \mathcal{S}_λ is the closed convex hull of \mathcal{M}_λ .

Iwanik uses these results to extend known results on the residuality of ergodic automorphisms in \mathcal{G} and \mathcal{M}_λ to \mathcal{S} and \mathcal{S}_λ . A. Ionescu Tulcea [IT] has shown that $\bigcup_{\lambda \sim \mu} \mathcal{M}_\lambda$ is meager in \mathcal{G} : in general, an invertible isometry preserves no measure. We show here that $\bigcup_{\lambda \sim \mu} \mathcal{S}_\lambda$ is meager in \mathcal{S} : in general, a stochastic operator preserves no measure. Combining this with another result of Iwanik [Iw], we can prove even more; the set \mathcal{S}_I of stochastic operators admitting a subinvariant measure absolutely continuous with respect to μ is also meager in \mathcal{S} : in general, a stochastic operator admits no subinvariant absolutely continuous measure. This extends Ionescu Tulcea's result to \mathcal{S} and in a sense complements and completes the Baire category results of Iwanik. Our method is to use the result of Iwanik on