

Algebras Generated by Analytic and Harmonic Functions

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In this paper we study algebras of functions in the open unit disk \mathbf{D} in the complex plane, and we find analogues of some well-known theorems about algebras of functions on the unit circle $\partial\mathbf{D}$. Let $L^\infty(\partial\mathbf{D})$ denote the usual space of bounded measurable functions on $\partial\mathbf{D}$, let $H^\infty(\mathbf{D})$ denote the algebra of bounded analytic functions on \mathbf{D} , and let $H^\infty(\partial\mathbf{D})$ denote the subalgebra of $L^\infty(\partial\mathbf{D})$ consisting of boundary values of functions in $H^\infty(\mathbf{D})$. A theorem of K. Hoffman ([3], p. 193), shows that if f is in $L^\infty(\partial\mathbf{D})$ but not in $H^\infty(\partial\mathbf{D})$, then the norm closed subalgebra of $L^\infty(\partial\mathbf{D})$ generated by $H^\infty(\partial\mathbf{D})$ and f contains $C(\partial\mathbf{D})$. Moving from the boundary to the interior, simple examples show that the norm closed algebra generated by $H^\infty(\mathbf{D})$ and a function $h \in L^\infty(\mathbf{D})$ need not contain $C(\mathbf{D}^-)$ (here \mathbf{D}^- denotes the closed unit disk); for example, take f to be zero on a proper open subset of \mathbf{D} .

In this paper we prove that if f is any bounded harmonic function on \mathbf{D} that is not analytic, then the norm closed algebra generated by $H^\infty(\mathbf{D})$ and f contains $C(\mathbf{D}^-)$. Since the Poisson integral is not multiplicative on $L^\infty(\partial\mathbf{D})$, the proof of Hoffman's result for the boundary algebra does not transfer to the interior. We also present a simplified version of W. Rudin's proof that $H^\infty(\mathbf{D}) + C(\mathbf{D}^-)$ is a closed subalgebra of $L^\infty(\mathbf{D})$.

The disk algebra A is defined to be the intersection of $H^\infty(\partial\mathbf{D})$ and $C(\partial\mathbf{D})$. The Wermer Maximality Theorem [9] states that if f is any function in $C(\partial\mathbf{D})$ that is not in A , then the norm closed algebra generated by A and f equals $C(\partial\mathbf{D})$. Passing from the boundary to the interior, we let $A(\mathbf{D}^-)$ denote the algebra of continuous functions on \mathbf{D}^- that are analytic on \mathbf{D} . Unlike the disk algebra A , the algebra $A(\mathbf{D}^-)$ is not a maximal subalgebra of $C(\mathbf{D}^-)$; more precisely, there is a function f in $C(\mathbf{D}^-)$ but not in $A(\mathbf{D}^-)$ such that the norm closed algebra generated by $A(\mathbf{D}^-)$ and f is not equal to $C(\mathbf{D}^-)$.

We will prove that if f is in $C(\mathbf{D}^-)$ and f is harmonic (but not analytic) on \mathbf{D} , then the norm closed algebra generated by $A(\mathbf{D}^-)$ and f equals $C(\mathbf{D}^-)$. A partial extension of this result to regions other than the disk is also obtained.