

Extreme Points of Certain Operator Algebras

R. L. MOORE & T. T. TRENT

The extreme points of the unit ball of a Banach space are always of interest; the Krein–Milman theorem alone provides sufficient motivation for identifying them. One possible application of a characterization of extreme points is in the study of isometries. Observe that if φ is a linear isometry of a Banach space X onto another one, Y , then the extreme points of the unit ball of X must be mapped by φ onto the unit extreme points of Y . Richard Kadison used precisely this observation in his study of isometries on C^* -algebras [3].

It was an interest in linear isometries of nest algebras of Hilbert space operators which led us to investigate extreme points for those algebras. In fact, the main results, Theorem 3 and its corollaries, can be applied to any norm closed subspace (and, in particular, any subalgebra) of Hilbert space operators. It is hoped that these results may be of use in investigations of many collections of operators. In this paper our focus is on nest algebras, although we indicate some other applications in Theorem 4 (Kadison's C^* -algebra result) and Theorem 5.

In what follows, \mathcal{H} represents a separable infinite-dimensional Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of bounded operators on \mathcal{H} , and \mathcal{B}_1 the unit ball in $\mathcal{B}(\mathcal{H})$: $\mathcal{B}_1 = \{T \in \mathcal{B}(\mathcal{H}) : \|T\| \leq 1\}$. If A is a bounded linear operator and if $\|A + B\| \leq \|A\|$ and $\|A - B\| \leq \|A\|$ for an operator B , an obvious convexity argument shows that $\|A \pm B\| = \|A\|$. In this case, we will say that B is a *perturbation* of A , or that B *perturbs* A . If $A \in \mathcal{B}_1$ and if A has any nonzero perturbation B , then A is not an extreme point of \mathcal{B}_1 , and an elementary argument proves the converse. Our goal is to show that if $\|A\| = 1$ and if A has a nonzero perturbation, then it also has one of the form $(1 - AA^*)^{1/2}X(1 - A^*A)^{1/2}$, for some X in $\mathcal{B}(\mathcal{H})$.

Lemma 1. *Suppose that $\|A\| = 1$ and that $\|A \pm B\| = 1$. Then there exist bounded operators S and T such that $B = S(1 - A^*A)^{1/2}$ and $B = (1 - AA^*)^{1/2}T$.*

Proof. We have

$$1 \geq (A + B)^*(A + B) = A^*A + B^*B + 2\operatorname{Re} A^*B \quad \text{and}$$

$$1 \geq (A - B)^*(A - B) = A^*A + B^*B - 2\operatorname{Re} A^*B.$$