

Equivalence of Slowly Decreasing Conditions and Local Fourier Expansions

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Let $\mathcal{E}_\omega(\mathbf{R})$ denote the Fréchet space of all ω -ultradifferentiable functions on \mathbf{R} in the sense of Beurling–Björck. We assume that the function $t \mapsto \omega(e^t)$ is convex on \mathbf{R} , define the weight function $p: \mathbf{C} \rightarrow [0, \infty[$ by $p(z) := |\operatorname{Im} z| + \omega(|z|)$ and denote by A_p the algebra of all entire functions f on \mathbf{C} which for some $A > 0$ and $B > 0$ satisfy the estimate

$$|f(z)| \leq A \exp(Bp(z)), \quad z \in \mathbf{C}.$$

Then it is well known that the Fourier–Laplace transform \wedge is an algebra isomorphism between the convolution algebra $(\mathcal{E}_\omega(\mathbf{R})'_b, *)$ and A_p . For $\mu \in \mathcal{E}_\omega(\mathbf{R})'$ one defines the convolution operator T_μ on $\mathcal{E}_\omega(\mathbf{R})$ by

$$T_\mu(\varphi): x \mapsto \langle \mu_y, \varphi(x - y) \rangle.$$

Extending results of Ehrenpreis [7] and Berenstein and Taylor [2] we prove that for $\mu \in \mathcal{E}_\omega(\mathbf{R})'$, the following assertions are equivalent:

- (1) T_μ is surjective,
- (2) $\hat{\mu}$ is slowly decreasing in A_p in the sense of Berenstein and Taylor [2],
- (3) there exists $A \geq 1$ such that for each $x \in \mathbf{R}$ there exists $t \in \mathbf{R}$ with $|t - x| \leq A\omega(x)$ and $|\hat{\mu}(t)| \geq \exp(-A\omega(t))$.

Then we use the estimates which have been derived in proving that (3) implies (2), together with an idea of Meise [14], to show that for a surjective convolution operator T_μ on $\mathcal{E}_\omega(\mathbf{R})$ with an infinite dimensional kernel, “local” zero solutions admit a local Fourier expansion. Roughly speaking, this means the following: There exists a Schauder basis $\{f_j\}_j$ of $\ker T_\mu$ consisting of exponential polynomials such that for each $\rho > 0$ there exist $r > \rho$ and $h_j \in \mathcal{E}_\omega[-r, r]'$, such that for each $R > r$ which is sufficiently large and each $\varphi \in \mathcal{E}_\omega[-R, R]$ with $T_\mu(\varphi)|_{[-r, r]} \equiv 0$, we have an expansion

$$\varphi(x) = \sum_{j=1}^{\infty} f_j(x) h_j(\varphi), \quad x \in [-\rho, \rho],$$