

The Notion of Formal Essential Finiteness for Smooth Real Hypersurfaces

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Introduction. One purpose of this paper is to generalize the notion of “essentially finite” for a real analytic real hypersurface germ in C^n to a smooth real hypersurface germ. In order to do so, we discover what pieces of the Taylor expansion of a smooth defining function for the hypersurface have an invariant meaning. As a bonus we obtain simple proofs that the conditions of “essentially finite” and its generalization, “formally essentially finite,” are nondegeneracy conditions. To do so, we give an upper semi-continuous integer valued function that measures the extent of essential finiteness (or formal essential finiteness). These methods also clarify the relationship of this finiteness criterion to other notions such as “points of finite type.”

We continue the introduction by recalling the theorem of Baouendi–Treves–Jacobowitz [BJT] on CR diffeomorphisms of real analytic hypersurfaces in C^n . We suppose that M and M' are real analytic, real hypersurfaces in C^n , and that F is a CR diffeomorphism between them. If each point of M satisfies the condition of essential finiteness, then F must be itself real analytic. We will give an alternate characterization of the notion of essential finiteness in Proposition 13, which we mention here. If (M, p) denotes the germ of M at p , we construct an ideal J_p of holomorphic germs at p from any analytic defining equation. Then (M, p) is essentially finite if and only if the ideal is primary to the maximal ideal m_p . For example, for the hypersurface M given by $\operatorname{Re}(z_n) + \sum |z_j|^{2p_j} = 0$, we assign the ideal $(z_1^{p_1}, \dots, z_{n-1}^{p_{n-1}}, z_n)$ to the origin, so that M is essentially finite there, and in fact everywhere.

This approach extends to the smooth case, by working with the formal power series rings $C[[z-p]]$ and $C[[z-p, \overline{z-p}]]$. For a smooth hypersurface germ (M, p) , we determine an invariant ideal J_p in the formal power series ring in the z_j (not including the conjugates). We define (M, p) to be “formally essentially finite” if J_p is m_p primary. In case (M, p) is real analytic, we obtain an ideal in the convergent power series ring (the ring of germs of holomorphic functions), and prove that this is m_p primary if and only if (M, p) is essentially finite according to the original definition in [BJT]. Finally, by considering the