

# *Conformal Deformation and the Heat Operator*

THOMAS P. BRANSON & BENT ØRSTED

**0. Introduction.** It has long been known that local geometric information on a pseudo-Riemannian manifold  $(M, g)$  is encoded in asymptotic expansions of the fundamental solutions of natural differential operators. An early exponent of this point of view was Hadamard [H], who worked with the (local) fundamental solution of the d'Alembertian on a Lorentz manifold. Hadamard's basic approach can also be applied to the Laplace-d'Alembert operators associated to metrics of any signature [C], and presumably to a wide class of operators built naturally from the metric. Minakshisundaram and Pleijel [MP] drew directly on Hadamard's ideas, especially the "transport equations", in their treatment of the fundamental solution of the heat operator on a compact Riemannian manifold.

The treatments of Hadamard and of Minakshisundaram and Pleijel have since been streamlined in two ways: by the introduction of distributions, especially in the case of the d'Alembertian ([Fr] is a good reference) and of pseudo-differential operators, especially for the heat operator ([Gi2] is a good reference). The "wave" and "heat" schools have grown apart, and in some cases calculations which are completely insensitive to the metric signature have been performed independently by both groups: compare [M], [Sa]. It is interesting that in the old transport formalism, it is easiest to get the fundamental solution  $H(t, x, y)$  of the heat operator first ( $t > 0$ ,  $x, y \in M$ ), and then integrate to get the (local) fundamental solution of the Laplace type operator  $D$ :

$$G(x, y) = \int_0^\infty H(t, x, y) dt$$

(formally,  $D^{-1} = \int_0^\infty \exp(-tD) dt$ ). In the pseudodifferential approach, one first constructs  $(D - \lambda)^{-1}$  (actually, its symbol in local coordinates) for  $\lambda$  a generic non-eigenvalue, and then builds  $\exp(-tD)$  using the Cauchy integral formula. Instead of working with two-point functions (i.e., locally in  $M \times M$ ) as in the transport approach, one works locally with functions on the cotangent bundle.

There has been much interest in calculating the local geometric invariants involved in the construction and asymptotic expansion of these fundamental solutions, in connection with, for example, the determination of necessary and