

A Model of Dislocations and the Associated Free Boundary Problem

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1. The dislocation problem. Dislocations occur during plastic deformation of metals and they move through the material causing plastic strain. This phenomenon has been studied by various authors; we refer to [5] [7] [8] [9] and the references given there. Recently, Head, Howison, Ockendon, Titchener and Wilmott [6] have proposed, in the two-dimensional case, a theory describing the motion of dislocations along parallel lines (to the x -axis). Their model is based on linear elasticity but with an inhomogeneous term which depends on the dislocation density $\omega(x,y)$; $\omega > 0$.

The stress at any point (x,y) due to the dislocations is $\nabla\varphi$ where

$$(1.1) \quad \varphi(x,y) = \frac{1}{2\pi} \iint \Omega_0 \omega(\xi,\eta) \log\{(x-\xi)^2 + (y-\eta)^2\}^{1/2} d\xi d\eta$$

and where Ω_0 is the region occupied by the dislocations. Let us denote by (σ, σ_1) the externally imposed stress field. Then the velocity of the dislocations at any point (x,y) is given by the vector $(\sigma + \varphi_x, 0)$. In the stationary case we then have

$$(1.2) \quad \sigma + \varphi_x = 0 \quad \text{in } \Omega_0.$$

From (1.1) we also get

$$(1.3) \quad \Delta\varphi = \omega \quad \text{in } \Omega_0$$

and

$$(1.4) \quad \Delta\varphi = 0 \quad \text{in } \Omega_1 \setminus \Omega_0$$

where Ω_1 is the region occupied by the metal. It is further assumed in [6] that

$$(1.5) \quad \varphi \quad \text{and} \quad \nabla\varphi \quad \text{are continuous across } \partial\Omega_0$$

and that the total number of dislocations along any line of motion $\Omega_{0,y} = \{x; (x,y) \in \Omega_0\}$ is a given function:

$$(1.6) \quad \int_{\Omega_{0,y}} \omega(x,y) dx = 2G(y).$$