

Inequalities for Logarithmic Coefficients of Univalent Functions and their Derivatives

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1. Introduction. Let S be the class of functions $f(z) = z + a_2z^2 + \cdots$ analytic and univalent in the unit disk \mathbf{D} . The logarithmic coefficients γ_n of f are defined by

$$\log \left[\frac{f(z)}{z} \right] = 2 \sum_{n=1}^{\infty} \gamma_n z^n.$$

Thus the Koebe function $k(z) = z(1-z)^{-2}$ has logarithmic coefficients $\gamma_n = \frac{1}{n}$. The inequality $|\gamma_n| \leq \frac{1}{n}$ holds for starlike functions $f \in S$, but is false for the full class, even in order of magnitude. (See [2], p. 242.)

In a series of papers in the 1960's, I. M. Milin drew attention to the logarithmic coefficients as a means for estimating the coefficients a_n of functions in S . Thus, for example, Milin's lemma

$$(1) \quad \sum_{k=1}^n k |\gamma_k|^2 \leq \sum_{k=1}^n \frac{1}{k} + \delta, \quad \delta < 0.312,$$

leads to the remarkable bound $|a_n| \leq e^{\delta/2} < 1.17$ for the coefficients of odd functions $f \in S$. (By more careful estimation, V. I. Milin recently improved the estimate to 1.14.) In particular, because of the failure of the Littlewood-Paley conjecture $|a_n| \leq 1$, the constant δ in (1) cannot be reduced to zero. However, Milin conjectured the inequality

$$(2) \quad \sum_{k=1}^n (n-k+1)k |\gamma_k|^2 \leq \sum_{k=1}^n (n-k+1) \frac{1}{k},$$

which he showed would imply the famous Bieberbach conjecture $|a_n| \leq n$. (See [5] or [2] for further details.) Subsequently, de Branges proved (2) and thus established the Bieberbach conjecture.

Duren and Leung [3] showed that

$$(3) \quad \sum_{k=1}^n k^2 |\gamma_k|^2 = O(n)$$