

Symmetry Diffeomorphism Group of a Manifold of Nonpositive Curvature, II

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Introduction. This paper is a continuation of [E5] in which we defined and studied the groups G^* and G_e^* of symmetry diffeomorphisms and even symmetry diffeomorphisms respectively of a complete, simply connected Riemannian manifold \widetilde{M} of nonpositive sectional curvature. In this paper \widetilde{M} will always denote such a manifold. Our main result is that there is a close correspondence, stated below, between orbits of the group G_e^* on the boundary sphere $\widetilde{M}(\infty)$ and orbits of the holonomy group Φ_p on the sphere $s_p\widetilde{M}$ of unit vectors at a point p of \widetilde{M} . Using a holonomy theorem due to Berger, the main result then yields a characterization of symmetric spaces of noncompact type and rank at least two in terms of the action of G_e^* on $\widetilde{M}(\infty)$, where \widetilde{M} is an arbitrary manifold. We emphasize that the elements of G^* and G_e^* are only diffeomorphisms of \widetilde{M} , and we do not assume that nonidentity isometries of \widetilde{M} exist. However, for applications of the main result and its corollary it is sometimes convenient to assume that the isometry group $I(\widetilde{M})$ is large in a sense made precise in 1.5 below. See for example, Theorem C later in the Introduction.

Given a point $p \in \widetilde{M}$, it is well known that the exponential map $\exp_p : T_p\widetilde{M} \rightarrow \widetilde{M}$ is a diffeomorphism. We define the geodesic symmetry $s_p : \widetilde{M} \rightarrow \widetilde{M}$ by

$$s_p = \exp_p \circ T \circ (\exp_p)^{-1},$$

where $T : \widetilde{SM} \rightarrow \widetilde{SM}$ sends v to $-v$ for all $v \in \widetilde{SM}$. Equivalently, the geodesic symmetry s_p is the unique diffeomorphism of \widetilde{M} such that $S_p(\gamma(t)) = \gamma(-t)$ for all $t \in \mathbf{R}$ and all geodesics γ of \widetilde{M} with $\gamma(0) = p$. We define the *symmetry diffeomorphism group* G^* to be that group of diffeomorphisms of \widetilde{M} generated by the geodesic symmetries $\{s_p : p \in \widetilde{M}\}$. The group G_e^* consisting of all products of an even number of geodesic symmetries is a normal subgroup of G^* with index two called the *even symmetry diffeomorphism group* of \widetilde{M} . Each geodesic symmetry extends to an involutive homeomorphism of the boundary sphere $\widetilde{M}(\infty)$ by defining $s_p(x) = \gamma_{px}(-\infty)$ for all $x \in \widetilde{M}(\infty)$, where we use the