

Uniform Differentiability of Sobolev Functions

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1. Introduction. In [MZ], the following Lusin-type approximation result was established for functions in the Sobolev space $W^{k,p}(\Omega)$.

Theorem 1.1. *Let j, k be positive integers with $j \leq k$, $(k-j)p < n$ and let Ω be a non-empty open set subset of \mathbf{R}^n . Then, for $u \in W^{k,p}(\Omega)$ and each $\varepsilon > 0$, there exists $v \in C^j(\Omega)$ such that*

$$B_{k-j,p}(F) < \varepsilon \quad \text{and} \quad \|u - v\|_{j,p;\Omega} < \varepsilon$$

where $F = \Omega \cap \{x : u(x) \neq v(x)\}$ and $B_{k-j,p}$ denotes Bessel capacity.

See Section 2 below for definitions and notation.

The proof of this theorem uses in a crucial way a result of [CZ, Theorem 8] which states that if a function $u \in L^p(\mathbf{R}^n)$ has a derivative of order k in the sense of L^p at all points of a compact set $E \subset \mathbf{R}^n$ (see Definition 2.1 below), then u is the restriction of a function $\bar{u} \in C^k(\mathbf{R}^n)$. It has recently come to our attention that this result is slightly in error, thus placing the validity of Theorem 1.1 in doubt. Indeed, an example in [WH] can be easily modified to show that the result of [CZ] is not true without additional hypotheses. In Theorem 3.2 it is shown that if we add the assumption that (2.3) holds uniformly on a compact set E , then u is the restriction of a $C^k(\mathbf{R}^n)$ function. In our context, the function u is taken as an element of $W^{k,p}(\mathbf{R}^n)$ and therefore it is known that u has an L^p -derivative of order j at all points except for a set of $B_{k-j,p}$ -capacity 0, for $j \leq k$, c.f. [BZ]. We will improve this result by showing that u satisfies (2.3) uniformly on the complement of sets of small $B_{k-j,p}$ -capacity (Theorem 3.5). This, along with Theorem 3.2 and other arguments in [MZ], leads to the conclusion that there is a function $v \in C^{k-j}(\mathbf{R}^n)$ which is close to u in the Sobolev norm and agrees with u everywhere except for a set of small capacity.