

# *Parabolic Measure and the Dirichlet Problem for the Heat Equation in Two Dimensions*

JOHN L. LEWIS & JUDY SILVER

**Introduction.** Let  $f$  be a continuous function on the real line,  $\mathbf{R}$ , with  $f(t) = 0$ ,  $|t| \geq 1$ , and put  $D = \{(x, t) : x > f(t)\}$ . If  $E \subseteq \mathbf{R}$  is Lebesgue measurable, let  $|E|$  be the Lebesgue measure of  $E$  and set

$$\rho(E) = \{(f(t), t) : t \in E\}.$$

Note that  $\rho(E)$  is contained in the boundary of  $D(\partial D)$ . If  $F \subseteq \partial D$  is a Borel set, let  $\omega^+(F)$  [ $\omega^-(F)$ ] denote the parabolic [adjoint parabolic] measure of  $F$  with respect to  $(10, 10)$  [ $(10, -10)$ ]. That is,  $\omega^+(F)$  [ $\omega^-(F)$ ] is the value at  $(10, 10)$  [ $(10, -10)$ ] of the solution to the heat [adjoint heat] equation in  $D$  with boundary value 1 on  $F$  and 0 on  $\partial D - F$  in the Brelot-Perron-Wiener sense.

In [20] Wu published the first of a series of papers (see also [21], [12], [13]) whose object was to find parabolic analogues of theorems proved by Dahlberg (see [2], [3], [4]) for the Laplacian in Lipschitz domains. She proved ([20, Theorem 1]) a theorem which for  $D$  can be stated as follows:

**Theorem A.** *Let  $f$ ,  $D$ , be as above and suppose for some  $A$ ,  $0 < A < \infty$ , that*

$$(1.1) \quad |f(t) - f(s)| \leq A|s - t|^{1/2}, \quad s, t \in \mathbf{R}.$$

*If  $E \subset [-1, 1]$  and  $|E| = 0$ , then there exists  $E^+$ ,  $E^-$ , with  $E = E^+ \cup E^-$  and*

$$\omega^+[\rho(E^+)] = \omega^-[\rho(E^-)] = 0.$$

We note that Theorem A and the maximum principle for the heat [adjoint heat] equation imply that  $\rho(E^+)$  [ $\rho(E^-)$ ] is of zero parabolic [adjoint parabolic] measure at each point in  $D$ . We also note that Kaufman-Wu in [13] showed that  $\omega^+ \circ \rho$ ,  $\omega^- \circ \rho$ , need not be absolutely continuous with respect to Lebesgue measure on  $[1, -1]$ . In fact they constructed an example of a function  $f$  satisfying (1.1) for which  $D = \{(x, t) : x > f(t)\}$  has the following property: There is a set  $E \subset [-1, 1]$  of Hausdorff dimension  $\alpha < 1$  such that  $\omega^\pm[\rho[-1, 1] - E] = 0$ .