

How to Recognize a Discrete Maximal Function

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Abstract. In a discrete setting, we characterize those positive functions which are maximal functions of positive measures. In particular, this applies to maximal functions of L^1 bounded martingales with respect to a p -adic family of σ -algebras, and more generally, to maximal functions on a tree.

1. Introduction and results. The origin of our problem is the following question. Given a measurable function $\omega > 0$ in \mathbf{R}^n , the map $f \rightarrow \omega^{-1}f$ gives a bijection between $L^1 = L^1(dx)$ and the weighted space L^1_ω . But it does not map the corresponding weak type space $L^{1,\infty}$ into its weighted analogue $L^{1,\infty}_\omega$. However, if M_0 is the Hardy-Littlewood maximal operator, it often happens that $\omega^{-1}M_0f \in L^{1,\infty}_\omega$ for all $f \in L^1$. Equivalently, we can take finite measures μ instead of f here. The question is now which ω have this property. From the definition, it can be seen that ω does not have the property precisely when ω is close to some $M_0\mu$, in a certain sense. That leads us to the problem of recognizing whether a given function is (close to) a maximal function. In this paper, we attack this problem by dealing with a discrete setting instead of \mathbf{R}^n . Bruna and Korenblum [1] have studied the related question of deciding whether a given function can be majorized by a maximal function.

We first describe our space. Let Ω be a measurable space with a positive σ -finite measure m . Instead of mE , we shall write $|E|$. We assume given a hierarchical system of partitions of Ω , as follows. For each $k \in \mathbf{Z}$, let Ω_k be the disjoint union of a finite or countably infinite family of measurable sets called pieces of order k . Each piece must have positive finite m -measure. Further, each piece of order k is the union of a finite number of pieces of order $k+1$. We assume that any decreasing sequence of pieces E_k , $k \geq k_0$, with E_k of order k , has a one-point intersection. (Here the essential part is that the intersection is nonempty.) One could always obtain the one-point property by passing to a quotient of Ω .) For convenience, we make the following "connectivity" assumption: for any two points of Ω , there is a piece containing both. (Without this assumption, Ω would break down into parts verifying the condition, which could be treated separately.) For each piece $E \neq \Omega$, it now follows that there exists a minimal piece \tilde{E} strictly containing E .