An Integral Inequality for Compact Spacelike Hypersurfaces in De Sitter Space and Applications to the Case of Constant Mean Curvature

SEBASTIÁN MONTIEL

The importance of maximal and constant mean curvature spacelike hypersurfaces in general relativity has been emphasized by a number of authors; see, for example, [CFM], [S], and references therein. These hypersurfaces are convenient as initial surfaces for the Cauchy problem in arbitrary spacetimes and for studying the propagation of gravitational radiation, respectively. Results concerning existence and uniqueness for this type of hypersurfaces are obtained in [B], [CB], [CFM], [N], [S], where various conditions are imposed on the ambient spacetime.

When the ambient spacetime is Lorentz–Minkowski space R_1^{n+1} and the spacelike hypersurface is given as a graph of a certain function u, the condition of constant mean curvature H is written in terms of u as follows:

$$(1-|\nabla u|^2)\Delta u+(\nabla^2 u)(\nabla u,\nabla u)=nH(1-|\nabla u|^2)^{3/2}, \qquad |\nabla u|^2<1,$$

where ∇, ∇^2 and Δ denote the usual gradient, Hessian and Laplacian of R^n respectively. If $H=0$, it was proved by Calabi [C] that, for $n\leq 4$, the only entire solutions to that equation are linear. Later, Cheng and Yau [CY] showed that the same holds for arbitrary n. The case $H\neq 0$, which has a completely different behaviour, was extensively studied by Treibergs in [T].

When the ambient is de Sitter space S_1^{n+1}, Goddard [G] conjectured that complete spacelike hypersurfaces with constant mean curvature H must be umbilical. Later, Akutagawa [A] has proved that Goddard's conjecture is true when $H^2<4(n-1)/n^2$, if $n>2$, and when $H^2\leq 1$, if $n=2$. Moreover, in that work

Indiana University Mathematics Journal ©, Vol. 37, No. 4 (1988)