

# *On Quasiconformal Mappings with Given Boundary Values*

JAMES A. JENKINS

*Dedicated to my teacher, Lars Ahlfors,  
on the occasion of his eightieth birthday*

1. The problem of characterizing the boundary values of a quasiconformal mapping from one hyperbolic simply-connected domain to another was solved by Beurling and Ahlfors [1]. Their solution, formulated in the context of mappings of a half-plane onto itself, was given in terms of so-called quasisymmetric functions. This solution is very useful for practical applications and almost elementary but conceptually has certain drawbacks. It is non-homogeneous with respect to the boundary elements. More important, it is hard to envisage a corresponding necessary and sufficient condition for the boundary values of a  $K$ -quasiconformal mapping (i.e., with a bound for the dilation specified in the conditions). There is indeed a second condition, almost trivially qualitatively equivalent to the preceding, in which the first objection is obviated. This operates with the ratios of modules of quadrangles determined by the domain with vertices corresponding under the given boundary mapping. A quite elaborate account of this is found in [4] where indeed the necessary and sufficient condition is formulated in this context. However, for the proof those authors fall back on the presentation of [1].

In this paper there is given a necessary and sufficient condition for the boundary values of a  $K$ -quasiconformal mapping in the above situation. Further, in the same context there is given a purely conceptual proof for the necessary and sufficient condition of [1], stated in terms of ratios of modules.

2. We consider two simply-connected domains of hyperbolic type  $D$  and  $D^*$  which for ease of formulation may be taken to be Jordan domains in the plane bounded by curves  $J$  and  $J^*$ . We will consider a sense-preserving homeomorphism  $\varphi$  of  $J$  onto  $J^*$  and ask for a necessary and sufficient condition for  $\varphi$  to be extendable to a  $K$ -quasiconformal mapping of  $D$  onto  $D^*$ . We choose on  $J$  points  $P_1, \dots, P_k$  in the positive sense. Teichmüller's basic theorem [5] characterizes the extremal quasiconformal mapping of  $D$  onto  $D^*$  whose extension to  $J$  and  $J^*$  relates  $P_j$  to  $\varphi(P_j)$ ,  $j = 1, \dots, k$ . In its full generality this is indeed a