

Asymptotic Behavior of C_0 -Semigroups in B -Convex Spaces

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1. Introduction. Suppose $A : X \supseteq D(A) \rightarrow X$ is the generator of a C_0 -semigroup $(U_A(t))_{t \geq 0}$ on a complex Banach space X . Let $\sigma(A; X)$ denote the *spectrum* of A and let $R(z, A) := (z - A)^{-1}$ be the resolvent defined on the resolvent set $\rho(A; X) = \mathbb{C} \setminus \sigma(A; X)$. If

$$(1.1) \quad s(A) := \sup\{\operatorname{Re}(z) : z \in \sigma(A; X)\}$$

denotes the *spectral bound* of A , we subdivide the half-plane $\operatorname{Re}(z) \geq s(A)$ according to the following growth abscissas of $R(\cdot, A)$: For $m \in \mathbb{N} \cup \{0\}$ let

$$(1.2) \quad s_m(A) := \inf\{s > s(A) : \|R(a + ib)\| = O(|b|^m) \text{ as } |b| \rightarrow \infty \text{ for } a \geq s\}$$

denote the *abscissa of m^{th} order growth* of $R(\cdot, A)$ along parallels to the imaginary axis.

In [9] M. Slemrod essentially proves the following:

Theorem 1.1. *Let $(U_A(t))_{t \geq 0}$ denote a C_0 -semigroup on a complex Banach space X and let $\alpha \in \rho(A; X)$. Then for every $\varepsilon > 0$, there is an $M_\varepsilon > 0$ such that*

$$(1.3) \quad \|U_A(t)R(\alpha, A)^{m+2}\| \leq M_\varepsilon e^{t(s_m(A) + \varepsilon)}.$$

Since $D(A^m) = R(\alpha, A)^m X$, (1.3) says that the growth abscissa of $R(\cdot, A)$ determines the asymptotic behavior of certain classical solutions of the Cauchy-problem associated with the semigroup. It is the purpose of the present paper to improve (1.3) as follows:

- (i) If X is a Hilbert space, then $m + 2$ can be replaced by m . For $m = 0$ this is an immediate consequence of Gearhart's theorem [3], and we mention that for $m \geq 0$ this result has also been obtained by G. Weiss [11] using different methods.
- (ii) If X is a B -convex space (for a definition see below), e.g., if X is uniformly convex, then $m + 2$ can be replaced by $m + 1$ and this will turn out to be the best possible constant outside of the class of Hilbert spaces. Consequently