

A PDE Approach to Geometric Optics for Certain Semilinear Parabolic Equations

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Introduction. This paper demonstrates a direct PDE approach to certain problems concerning “geometric optics” approximations for solutions of certain nonlinear reaction-diffusion equations containing a small parameter. Such equations arise in physics, chemical kinetics and biology: see for example Aronson and Weinberger [1], [2], Bramson [5], Smoller [22], etc. Our work is mainly inspired by M. Freidlin’s investigations ([15], [16], [17]) of such equations using large deviation probabilistic techniques.

A model problem illustrating the type of problem addressed by Freidlin and ourselves is the following. Fix $D > 0$, and consider then the Kolmogorov-Petrovskii-Piskunov (KPP) equation (Fisher [11], Kolmogorov, Petrovskii and Piskunov [20]):

$$(0.1) \quad \begin{cases} u_t = \frac{D}{2} u_{xx} + u(1-u) & \text{in } \mathbf{R} \times (0, \infty), \\ u = \chi_{(-\infty, 0]} & \text{on } \mathbf{R} \times \{t = 0\}, \end{cases}$$

where $\chi_{(-\infty, 0]}$ is the indicator function of the set $(-\infty, 0]$. We are interested in the convergence of u to its equilibrium states 0,1 for large x and t . We thus introduce the scaling

$$u^\varepsilon(x, t) \equiv u\left(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}\right)$$

for $\varepsilon > 0$. Then Equation (0.1) becomes

$$(0.2)_\varepsilon \quad \begin{cases} u_t^\varepsilon = \frac{\varepsilon D}{2} u_{xx}^\varepsilon + \frac{1}{\varepsilon} u^\varepsilon(1-u^\varepsilon) & \text{in } \mathbf{R} \times (0, \infty), \\ u^\varepsilon = \chi_{(-\infty, 0]} & \text{on } \mathbf{R} \times \{t = 0\}, \end{cases}$$

and we inquire as to the limit behavior of u^ε as $\varepsilon \rightarrow 0$. Since

$$u^\varepsilon(1-u^\varepsilon) = \varepsilon u_t^\varepsilon - \frac{\varepsilon^2 D}{2} u_{xx}^\varepsilon,$$