

Elliptic Boundary Value Problems with Singular Coefficients and Critical Nonlinearities

HENRIK EGNELL

Introduction. In this paper we shall give some existence and nonexistence results for the boundary value problem:

$$(0.1) \quad \begin{cases} -\Delta_m u - |x|^\nu u^p = \lambda u^{m-1} & \text{in } \Omega \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Here Δ_m is the m -Laplace operator defined by $\Delta_m u = \nabla \cdot (|\nabla u|^{m-2} \nabla u)$, $1 < m < n$, $\nu > -m$, Ω is a bounded smooth domain containing the origin and we have the critical exponent: $p + 1 = \frac{m(n+\nu)}{n-m}$.

By critical exponent, we mean that we choose p so that the imbedding

$$H_0^{1,m}(\Omega) \hookrightarrow L^{p+1}(\Omega, |x|^\nu dx),$$

is continuous but not compact (see Lemma 7 and 10 below). This imbedding is not continuous if ν is positive. If we take Ω to be a ball $B_1(0)$ with radius 1 and centered at the origin, and replace $H_0^{1,m}(\Omega)$ by

$$\mathcal{R}_0^{1,m} = \{u \in H_0^{1,m}(B_1(0)) : u \text{ is radial}\},$$

the new imbedding will be continuous. Therefore, if ν is positive we shall only consider radial solutions of (0.1).

This type of problem has been studied in several papers. Let us mention some of them. The case when $m = 2$ and $\nu = 0$ was first treated in a famous paper by Brezis and Nirenberg [BN]. Some of their results have been generalized by Escobar [ES] and the author [EG1], to include the case with variable nonsingular coefficients. The generalization to general m has been carried out by Guedda and Veron [GV] and by the author [EG2]. Atkinson, Peletier and Serrin [APS] have studied radial solutions of (0.1), when $\nu = 0$. They develop