

On The Existence of Value Functions of Two-Player, Zero-Sum Stochastic Differential Games

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Introduction. In this paper we investigate the question of the existence of the value of zero-sum, two-player stochastic differential games. We formulate a notion of value functions (upper and lower) which satisfy the dynamic programming principle and are the unique (viscosity) solutions of the associated Bellman-Isaacs partial differential equations. In the case that the Isaacs' condition holds this implies that the game has a value. Finally, we establish the convergence of fully discrete approximations to the upper and lower value functions. For a general introduction to the subject of differential games we refer to R. Isaacs [15]. For a more expanded discussion of the scope of the theory as well as a brief description of the results presented here we refer to W. H. Fleming and P. E. Souganidis [12].

In what follows for the sake of definiteness we consider the case of a finite-horizon stochastic differential game, which in the sequel will be referred to as (SDG), with the state variable in \mathbf{R}^N and horizon $T > 0$. Other cases (e.g., state variable in bounded domains, infinite horizon problems, etc.) can be treated in a similar way. The (SDG) consists of the *dynamics*, i.e., a stochastic differential equation

$$(0.1) \quad dX_s = f(s, X_s, Y_s, Z_s) ds + \sigma(s, X_s, Y_s, Z_s) dW_s \quad (s \in [t, T]),$$

with initial condition

$$(0.2) \quad X_t = x \quad (x \in \mathbf{R}^N),$$

and the *pay-off*,

$$(0.3) \quad J(x, t; Y, Z) = E_{xt} \left\{ \int_t^T h(s, X_s, Y_s, Z_s) ds + g(X_T) \right\}.$$

Here W is a standard M -dimensional brownian motion, Y and Z are stochastic processes taking values in some compact subsets \mathcal{Y} and \mathcal{Z} or \mathbf{R}^p and \mathbf{R}^q respectively, for some positive integers p and q . Precise assumptions on Y and