

A Bootstrap Argument for Grad Generalized Differential Equations

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1. Introduction. In this work we study regularity theory for the Grad generalized differential equation:

$$\Delta\psi = -\nu(\psi)\psi^{*\prime\prime}(\mu_\psi(\psi)) - \frac{1}{2}\nu'(\psi)\psi^{*\prime 2}(\mu_\psi(\psi))$$

and minimizers of the associated Grad functional:

$$\int |\nabla\psi|^2 + \int \nu(\psi^*)\psi^{*\prime 2}.$$

Here ψ^* is the increasing rearrangement of ψ , $\mu(t) = |\{\psi < t\}|$, and ν is a prescribed function.

Roughly speaking, our basic result is that if a solution or minimizer ψ is initially assumed to have some regularity ($W^{2,2}$ to $C^{2,1}$, depending on the particular case) and $|\nabla\psi| > 0$, then ψ is infinitely differentiable. We also discuss cases when the assumption $|\nabla\psi| > 0$ can be dropped, for example when the underlying domain is convex and has certain symmetries.

One should note that Grad generalized differential equations (also known as queer differential equations) are non-linear, nonlocal, functional differential equations. This is due to the appearance of the rearrangement ψ^* and its first and second derivatives. They are also elliptic in nature.

These equations arise in modelling quasi-static equilibrium of confined plasmas in axisymmetric toroidal magnetic confinement devices. We refer the reader to [6, 7, 8] for an introduction to the physical motivation of this model and to [15] for a survey of the related mathematical literature.

In the natural physical setting of the problem the domain in which one wishes to solve these equations is a two dimensional cross section of a torus (in physicist's terms a poloidal cross section). In this paper, we will consider the problem in bounded subdomains of \mathbf{R}^n since this does not introduce any significant additional complications.

H. Grad conjectured based on extensive numerical evidence that solutions, for convex domains and analytic positive profile function ν (see [15]), are analytic. Motivated by this we initiated in [14] a program to demonstrate the regularity of