

Rank-2 Vector Bundles on a Surface with Many Spread Sections

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In [B] we introduced the following numerical invariant $s(E)$ for a vector bundle E on an integral variety V , $\dim(V) = \text{rank}(E)$, E spanned by its global sections. This invariant measures how many and how spread (from a birational point of view) are the sections of E . Set $n := \dim(V)$, $t := c_n(E)$, and assume $t > 0$. Assume (as always in this paper) that we work over an algebraically closed field \mathbf{F} with $\text{char}(\mathbf{F}) = 0$. By Bertini's theorem there is a dense open subset U of $H^0(V, E)$ such that, for every $s \in U$, the locus of zeroes $(s)_0$ is formed by t smooth points of V . We obtain a morphism $g : U \rightarrow S^t(V)$ (symmetric power). Set $s(E) := \dim(\text{Im}(g))$. Since E is spanned, we have $n \leq s(E) \leq nc_n(E)$. Recently ([LS], [W], [B], [L]) many authors were interested in the classification of ample spanned vector bundles with low invariants. In [B] we considered this problem, for instance, when $n = 2$, c_2 is low and $s(E)$ is maximal or minimal; sometimes we did not need the assumption “ E ample”, sometimes we needed the assumption “ $\det(E)$ very ample”. Now we consider only the case $n = 2$. We fix a smooth, connected, projective surface S and rank-2 vector bundle E on S , E spanned by its global sections. In Section 1 we consider a few properties of (S, E) under the assumption that $s(E) = 2c_2(E) - 1$ and $c_2(E) > 2$; for instance, we show (1.1) that S is rational. In Section 2 we classify the ample spanned E with $\det(E)$ very ample, $s(E) = 2c_2(E)$, and $4 \leq c_2(E) \leq 7$. In Section 5 we do the same when $s(E) = 2c_2(E) - 1$. In Section 4 we show that there is no ample spanned E with $c_2(E) = 3$, $s(E) = 5$. In Section 3 we classify spanned vector bundles E with $c_2(E) = 7$ and $s(E) = 14$. In Section 6 we prove a few results for bundles with $s(E) = 2c_2(E) - 2$: if $c_2(E) \geq 4$, $q(S) = 0$.

0. Notations. A sheaf on a projective variety is called spanned if it is generated by its global sections. We will write \mathcal{O} , K (the canonical line bundle) when there is no danger of misunderstanding. We will often consider a line bundle M on a smooth surface Y as a (class of a) divisor on Y ; in particular, we will write $|M|$ for the associated complete linear system and $M \cdot N$ or MN for the intersection numbers of two line bundles or divisors. Let $c : S \rightarrow X$ be a birational