

Construction of Bundles on Surfaces with Low c_2

EDOARDO BALLICO

In this paper we show (in a concrete example) how to check (if lucky) that a suitable possible bundle exists, is ample, is spanned by global sections, and its sections are “spread,” or that one of these conditions fails (see Section 3 (iii)). As an application we give (see Theorem 0.1) a “reasonably complete and explicit” classification of all pairs (S, E) with the following properties: S is a smooth, projective surface (over an algebraically closed field with zero characteristic), E is a rank-2 ample vector bundle on S spanned by its global sections and with $\det(E)$ very ample, $c_2(E) = 4$ and $s(E) = 6$. We recall the meaning of the invariant $s(E)$. Fix an integral projective variety of dimension n and a rank- n vector bundle F on V spanned by its global sections; set $t := c_n(F)$ and assume $t > 0$. By Bertini’s theorem there is an open dense subset U of $H^0(V, F)$ such that, for every $s \in U$, the zero locus $(s)_0$ is formed by t distinct smooth points. We get a morphism $g : U \rightarrow S^t(V)$; set $s(F) := \dim(\operatorname{Im}(g))$.

Here is the main result of this paper.

Theorem 0.1. *Let S be a smooth, connected, projective surface and E a rank-2 ample vector bundle spanned by its global sections. Set $L := \det(E)$ and assume L very ample, $c_2(E) = 4$ and $s(E) = 6$. Then either S is a smooth cubic surface in P^3 or S is a smooth complete intersection of two quadrics in P^4 . In both cases $L \simeq \mathcal{O}_S(2)$; in the last case $E \simeq 2\mathcal{O}_S(1)$.*

Viceversa, even in the case of cubic surfaces, for every such smooth S the family of bundles E as in Theorem 0.1 is not empty; these bundles are described in Section 3 (iii); counting dimensions, one sees that if $\deg(S) = 3$, there is a one-dimensional family of such E .

In Section 1 we prove some results on (S, E) as in the hypothesis of 0.1 under the assumption that $K_S \otimes L$ is very ample (e.g., we prove that S is rational). This section is a continuation of [Ba], Section 6. In Section 2 we prove that no such (S, E) exists if $K_S \otimes L$ is not ample; here we use the classification in [SV]. In Section 3 we conclude the proof, using the classification in [Li].

Notations. A vector bundle on a projective variety is said to be spanned if it is generated by its global sections. We will write \mathcal{O} , K (the canonical line