

Solutions of Matukuma's Equation with Finite Total Mass

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In 1930 the astrophysicist Matukuma [2] developed a model for a globular stellar system in which the gravitational potential u (positive) of the system satisfies the semilinear elliptic differential equation

$$(1) \quad -\Delta u = \frac{u^\gamma}{1 + |x|^2}, \quad x \in \mathbf{R}^3, \gamma > 1.$$

If $1 < \gamma < 5$, our results imply, in particular, that (1) has a positive radial solution $u \in C_{\text{loc}}^2(\mathbf{R}^3)$ such that $u(x) = O(|x|^{-1+\varepsilon})$ as $|x| \rightarrow \infty$ for arbitrary $\varepsilon > 0$. This implies easily the finiteness of the total mass

$$m_\gamma(u) = -\frac{1}{4\pi} \int_{\mathbf{R}^3} (\Delta u)(x) dx.$$

Indeed, $\gamma(-1 + \varepsilon) < -1$ with a choice $0 < \varepsilon < 1 - \frac{1}{\gamma}$, from which $m_\gamma(u)$ is finite by (1).

For $\gamma = 3$, Matukuma [3] found the exact solution $u(x) = \sqrt{3}(1 + |x|^2)^{-1/2}$, with finite total mass. Recently Ni and Yotsutani [5] proved, in particular, that (1) has a positive radial solution u in \mathbf{R}^3 with infinite total mass whenever $u(0)$ is sufficiently small, $1 < \gamma < 5$. Furthermore, they showed that no positive radial solution with finite total mass can exist if $\gamma \geq 5$. One of our objectives is to prove that such a solution does exist if $1 < \gamma < 5$, partially, affirming a conjecture in [5].

Our result, stated above for (1), will be generalized to the elliptic equation

$$(2) \quad -\Delta u = p(|x|)u^\gamma, \quad x \in \mathbf{R}^N, N \geq 3,$$

where $p \in C_{\text{loc}}^\alpha(\mathbf{R}_+)$ for $\alpha \in (0,1)$, $0 \not\equiv p(r) \geq 0$, $p(r) = O(r^{-a})$ as $r \rightarrow \infty$, $a > 0$, and

$$(3) \quad \begin{cases} 1 < \gamma < \frac{N+2}{N-2} & \text{if } a \geq 2, \\ \frac{N+2-2a}{N-2} < \gamma < \frac{N+2}{N-2} & \text{if } 0 < a \leq 2. \end{cases}$$

Theorem. For N , a , and γ as in (3), equation (2) has a positive radial solution $u \in C_{\text{loc}}^2(\mathbf{R}^N)$ with finite total mass.