

The Bloch Constant of Bounded Harmonic Mappings

FLAVIA COLONNA

ABSTRACT. In this work we observe that a harmonic mapping h of the open unit disk Δ into itself is Lipschitz when regarded as a function of the hyperbolic disk (Δ, ρ) in \mathbf{C} , endowed with the euclidean distance, and its Lipschitz number

$$\beta_h = \sup_{z \neq w} \frac{|h(z) - h(w)|}{\rho(z, w)},$$

called the Bloch constant of h , does not exceed $\frac{4}{\pi}$. Writing $h = f + \bar{g}$, with f and g analytic, we show that

$$\beta_h = \sup(1 - |z|^2)[|f'(z)| + |g'(z)|],$$

generalizing a well-known formula for analytic functions. We then characterize the extremal ones: $\beta_h = \frac{4}{\pi}$ if and only if there exists a constant ζ of modulus 1 and an analytic function $g : \Delta \rightarrow \Delta$ such that $\operatorname{Re}(\zeta h) = \frac{2}{\pi} \operatorname{Arg}\left(\frac{1+g}{1-g}\right)$, where g has Lipschitz number 1 as a map from (Δ, ρ) to itself. Furthermore the imaginary part of ζh need not be the constant 0, unless g is an inner function.

Throughout this paper we shall consider harmonic mappings with domain Δ , the open unit disk. A *harmonic mapping* is a function of the form $h = u + iv$, where u and v are real-valued harmonic functions. In particular, a harmonic mapping may also be represented in the form $h = f + \bar{g}$, where f and g are analytic functions. This representation is unique up to an additive constant.

Definition. Let h be a harmonic mapping satisfying the Lipschitz condition when regarded as a function from the hyperbolic disk into \mathbf{C} , endowed with the euclidean distance. The function h is called *Bloch* and the Lipschitz number

$$\beta_h = \sup_{z \neq w} \frac{|h(z) - h(w)|}{\rho(z, w)}$$